

HÖLDER CONTINUITY OF A BOUNDED WEAK SOLUTION OF GENERALIZED PARABOLIC p -LAPLACIAN EQUATIONS

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ABSTRACT. Here we generalize quasilinear parabolic p -Laplacian type equations to obtain the prototype equation as

$$u_t - \operatorname{div}(g(|Du|)/|Du| \cdot Du) = 0,$$

where a nonnegative, increasing, and continuous function g trapped in between two power functions $|Du|^{g_0-1}$ and $|Du|^{g_1-1}$ with $1 < g_0 \leq g_1 < \infty$. Through this generalization in the setting from Orlicz spaces, we provide a uniform proof with a single geometric setting that a bounded weak solution is locally Hölder continuous considering $1 < g_0 \leq g_1 \leq 2$ and $2 \leq g_0 \leq g_1 < \infty$ separately. By using geometric characters, our proof does not rely on any of alternatives which is based on the size of solutions.

INTRODUCTION

In 1957, DeGiorgi [4] showed that bounded weak solutions of linear elliptic partial differential equations are Hölder continuous, and his method was used by Ladyzhenskaya and Ural'tseva in [15] to show that bounded weak solutions of the quasilinear elliptic equation

$$\operatorname{div} \mathbf{A}(x, u, Du) = 0$$

are Hölder continuous if there are positive constants $p > 1$, C_0 , and C_1 such that

$$\mathbf{A}(x, u, \xi) \cdot \xi \geq C_0 |\xi|^p, |\mathbf{A}(x, u, \xi)| \leq C_1 |\xi|^{p-1}$$

for all $\xi \in \mathbb{R}^n$, where n is the number of space dimensions. (The theorem of De Giorgi is really just the case $p = 2$ here.) For parabolic equations

$$(0.1) \quad u_t - \operatorname{div} \mathbf{A}(x, t, u, Du) = 0,$$

Ladyzhenskaya and Ural'tseva followed De Giorgi's method with some modifications but they were only able to prove Hölder continuity under the structure conditions

$$(0.2) \quad \mathbf{A}(x, t, u, \xi) \cdot \xi \geq C_0 |\xi|^p, |\mathbf{A}(x, u, \xi)| \leq C_1 |\xi|^{p-1}$$

when $p = 2$.

There was little progress on the Hölder continuity of solutions when $p \neq 2$ until 1986, when DiBenedetto [6] proved the Hölder continuity result for $p > 2$. A key new step was his introduction of the concept of intrinsic scaling, which has since become an important aspect in the theory and which is discussed at great length in [22]. It took several more years until the joint work of Chen and DiBenedetto [2, 3] showed that bounded weak solutions are Hölder continuous also for $p < 2$. Unfortunately, these proofs are quite technical and their exposition (for example, in Chapters III and IV of [8]) is quite long. More recently, Gianazza, Surnachev, and Vespri [12] developed a more geometric approach to the Hölder continuity of solutions to equations when $p > 2$; their proof is simpler and more natural than the original one, but several issues from that proof still remain that we address here.

The more important ones are related to the distinction between the cases $p > 2$ and $p < 2$. All previously published proofs of Hölder continuity have treated these cases separately because of different qualitative behavior of solutions in the two cases. For example, any nonnegative solution of (0.1) which vanishes at a point (x_0, t_0) also vanishes in any cylinder with top center point (x_0, t_0) if $p \geq 2$; however, when $p < 2$, nonnegative solutions generally become zero in finite time. (We refer the reader to Sections VI.2, VII.2, and VII.3 of [8] for a more complete discussion of these phenomena.) Such behavior must be accounted for, and our proof finds a way to do so without considering the two cases separately. A further issue is that the newer proofs (see the Remark on page 278 of [12] for the case $p > 2$ and Section 4 of [11] for a related result in case $p < 2$) give a Hölder exponent which degenerates as p approaches 2; in both cases, the proof must be further modified for p close to 2 if the Hölder exponent is to remain positive near $p = 2$ even though the original proofs of Hölder continuity (in [2, 3, 6]) allowed a stable Hölder exponent in this case.

In this paper, we take a more general approach to the problem: We study (0.1) when there is an increasing function g such that

$$(0.3a) \quad \mathbf{A}(x, t, u, Du) \cdot Du \geq C_0 G(|Du|),$$

$$(0.3b) \quad |\mathbf{A}(x, t, u, Du)| \leq C_1 g(|Du|)$$

for some positive constants C_0 and C_1 , where G is defined by

$$G(\sigma) = \int_0^\sigma g(s) ds,$$

and we assume that there are constants g_0 and g_1 satisfying $1 < g_0 \leq g_1 < \infty$ such that

$$(0.4) \quad g_0 G(\sigma) \leq \sigma g(\sigma) \leq g_1 G(\sigma)$$

for all $\sigma > 0$. (The two inequalities in (0.4) are known as Δ_2 and ∇_2 conditions in Orlicz space theory as in Section I.3 & I.4 of [14] and in Section 2.3 of [21].) The structure (0.2) is contained in this model as the special case $g(s) = s^{p-1}$, in which case we may take $g_0 = g_1 = p$. In addition, our structure allows consideration of more general equations; as shown on pages 313 and 314 of [18], for any α and β with $1 < \alpha < \beta < \infty$, we can find a function g satisfying (0.4) such that

$$\limsup_{s \rightarrow \infty} \frac{g(s)}{s^\beta} > 0, \quad \liminf_{s \rightarrow \infty} \frac{g(s)}{s^\alpha} < \infty,$$

so we consider a class of structure functions g much wider than that of just power functions. In this way, we obtain a uniform proof of Hölder continuity (with appropriate uniformity of constants) for all $p \in (1, \infty)$ at once under the structure condition (0.2) as well as a proof of Hölder continuity under more general structure conditions. We note especially that the exponent is uniformly controlled (assuming (0.2)) over any finite range of p that stays away from 1, so our result in this case is stable as p approaches 2.

We point out here that the motivation for considering (0.4) comes from [18] in which corresponding results for elliptic equations were proved. The extension of the methods used in [18] for proving Hölder continuity of weak solutions to parabolic equations is not straightforward, and this paper presents the only such extension known to the authors.

For the extension, we also need a suitable definition of weak solution, and we present it here. For an arbitrary open set $\Omega_T \subset \mathbb{R}^{n+1}$, we introduce the generalized Sobolev space $W^{1,G}(\Omega_T)$, which consists of all functions u defined on Ω_T with weak derivative Du satisfying

$$\iint_{\Omega_T} G(|Du|) \, dx \, dt < \infty.$$

We say that $u \in C_{\text{loc}}(\Omega_T) \cap W^{1,G}(\Omega_T)$ is a weak subsolution of (0.1) if

$$0 \geq - \iint_{\Omega_T} u \varphi_t \, dx \, dt + \iint_{\Omega_T} \mathbf{A}(x, t, u, Du) \cdot D\varphi \, dx \, dt$$

for all $\varphi \in C^1(\bar{\Omega}_T)$ which vanish on the parabolic boundary of Ω_T ; a weak supersolution is defined by reversing the inequality. In fact, we shall use a larger class of φ 's which we discuss in a later section.

Our method of proof uses some recent ideas of Gianazza, Surnachev, and Vespri [12], who gave a different proof for the Hölder continuity in [1, 6]. While [1, 6] examine an alternative based on the size of the set on which $|u|$ is close to its maximum, the method in [12] use a geometric approach from regularity theory and Harnack estimates. We shall not discuss the Harnack estimate here, but the geometry from [12] is an important ingredient of our proof. On the other hand, [12] takes advantage of the nonvanishing of nonnegative solutions of degenerate equations for all time, so we need to use some ideas from [2, 3] to analyze the corresponding behavior of more general equations.

The proof is based on studying two cases separately. Either a bounded weak solution u is close to its maximum at least half of a cylinder around (x_0, t_0) or not. In either case, the conclusion is that the essential oscillation of u is smaller in a subcylinder centered at (x_0, t_0) . Basically, our goal is reached using geometric characters of u with two integral estimates, local and logarithmic estimates (4.1), (4.9).

In Section 1, we provide some preliminary results, mostly involving notation for our geometric setting. Section 2 states the main lemma and uses that lemma to prove the Hölder continuity of the weak solutions. The main lemma is proved in Section 3, based on some integral inequalities which are proved in Section 4.

1. PRELIMINARIES

Notation.

- (1) The set of parameters $\{g_0, g_1, N, C_0, C_1\}$ are the data.
- (2) Let K_ρ^y denote the N -dimensional cube centered at $y \in \mathbb{R}^N$ with the side length 2ρ , i.e.,

$$K_\rho^y := \left\{ x \in \mathbb{R}^N : \max_{1 \leq i \leq N} |x_i - y_i| < \rho \right\}.$$

For simpler notation, let $K_\rho := K_\rho^0$.

- (3) For given $(x_0, t_0) \in \mathbb{R}^{N+1}$ and positive constants r and s , we name a cylinder

$$\Omega_{r,s}^{x_0,t_0} := K_r^{x_0} \times [t_0 - s, t_0]$$

to refer any arbitrarily given cylinder. In addition, introduce three constants m , M , and ω such that

$$m \leq \operatorname{ess\,inf}_{\Omega_{r,s}^{x_0,t_0}} u(x, t),$$

$$M \geq \operatorname{ess\,sup}_{\Omega_{r,s}^{x_0,t_0}} u(x, t),$$

$$\omega \geq \operatorname{ess\,osc}_{\Omega_{r,s}^{x_0,t_0}} u(x, t).$$

- (4) For given $(x_0, t_0) \in \mathbb{R}^{N+1}$, given positive constants ρ and k , and a positive constant θ to be determined later depending on data, we say

$$\begin{aligned} T_{k,\rho} &:= \theta k^2 G \left(\frac{k}{\rho} \right)^{-1}, \\ Q_{k,\rho}^{x_0,t_0} &:= K_\rho^{x_0} \times [t_0 - T_{k,\rho}, t_0], \\ Q_{k,\rho} &:= Q_{k,\rho}^{0,0}. \end{aligned}$$

Note that $\rho > 0$ and $k > 0$ in both $T_{k,\rho}$ and $Q_{k,\rho}$ are replaceable. Details for $T_{k,\rho}$ and θ is following.

Geometry. The local energy estimate (4.1) plays crucial role in this paper which is nonhomogeneous unless $g_0 = g_1 = 2$. By controlling the length of time axis, we make two competing terms in (4.1) equivalent; that is, find $T_{k,\rho}$ from

$$G^{r-1} \left(\frac{\omega}{\rho} \right) \omega^{s+2} \frac{1}{T_{k,\rho}} \sim G^r \left(\frac{\omega}{\rho} \right) \omega^s,$$

for some constants r and s which directly leads to our definition of $T_{k,\rho}$.

This idea is so called intrinsic scaling introduced by DiBenedetto [8, 22]; roughly speaking, a weak solution of parabolic p -Laplacian type equation behaves like a solution of the heat equation in an intrinsically scaled cylinder. To reflect different natures of degenerate and singular equations, original proof by DiBenedetto [6] and Chen and DiBenedetto [2, 3] applied intrinsic time scaling for degenerate equation ($p > 2$) and intrinsic side length scaling for singular equation ($1 < p < 2$), respectively.

To carry a uniform geometric setting, we introduce the constant $\theta > 0$ depending on data reflecting behavior of a weak solution which is explicitly determined in Section 3. The constant θ is adjusted to capture two behaviors of solutions. First, when $g_0 < 2$, the constant θ is chosen to avoid reaching T^* where a weak solution may become extinct.

Second, when $g_1 > 2$, the constant θ is determined to guarantee enough time length so positive information can be extended over the cube.

Now, suppose that u is a bounded weak solution of (0.1) under (0.3) in $\Omega_{r,s}^{x_0,t_0}$. Then first choose $R > 0$ such that

$$(1.1) \quad 4R \leq \min \left\{ r, \frac{\omega}{G^{-1}(\theta\omega^2s^{-1})} \right\}$$

which implies

$$Q_{\omega,4R}^{x_0,t_0} \subset \Omega_{r,s}^{x_0,t_0}.$$

Without loss of generality, we let $(x_0, t_0) = (0, 0)$. For any arbitrary given cylinder, we can fit in the cylinder $Q_{\omega,4R}$ by selecting R properly. Basically, we are going to work with the cylinder $Q_{\omega,4R}$ to find a proper subcylinder where a solution has less oscillation eventually leading to Hölder continuity.

Useful inequalities. Because of generalized function g and G , we are not able to apply Hölder inequality or typical Young's inequality. Here we deliver essential inequalities which will be used through out the paper.

Lemma 1.1. *For a nonnegative and nondecreasing function $g \in C[0, \infty)$, let G be the antiderivative of g . Suppose that g and G satisfies (0.4). Then for all nonnegative real numbers σ , σ_1 , and σ_2 , we have*

- (a) $G(\sigma)/\sigma$ is a monotone increasing function.
- (b) For $\beta > 1$,

$$\beta^{g_0}G(\sigma) \leq G(\beta\sigma) \leq \beta^{g_1}G(\sigma).$$

- (c) For $0 < \beta < 1$,

$$(1.2) \quad \beta^{g_1}G(\sigma) \leq G(\beta\sigma) \leq \beta^{g_0}G(\sigma).$$

- (d) $\sigma_1g(\sigma_2) \leq \sigma_1g(\sigma_1) + \sigma_2g(\sigma_2)$.

- (e) (Young's inequality) For any $\epsilon > 0$,

$$\sigma_1g(\sigma_2) \leq \epsilon^{1-g_1}g_1G(\sigma_1) + \epsilon g_1G(\sigma_2).$$

Proof. This lemma is quoted directly or modified from the Lemma 1.1 from [18].

- (a) For $\sigma > 0$, due to the left hand side inequality of (0.4), we easily obtain

$$\frac{d}{d\sigma} \left(\frac{G(\sigma)}{\sigma} \right) = \frac{\sigma g(\sigma) - G(\sigma)}{\sigma^2} \geq (g_0 - 1) \frac{G(\sigma)}{\sigma^2} > 0$$

because $g_0 > 1$.

(b) The left inequality of (0.4) gives

$$\frac{g_0}{\xi} \leq \frac{g(\xi)}{G(\xi)} \text{ for } \xi \in (0, \infty).$$

By taking the integral over σ to $\beta\sigma$, we obtain

$$g_0 \log \frac{\beta\sigma}{\sigma} \leq \log \frac{G(\beta\sigma)}{G(\sigma)}$$

which implies

$$\beta^{g_0} G(\sigma) \leq G(\beta\sigma).$$

Similar argument with the right hand side of (0.4) completes the proof.

(c) Like the proof for (b), but take integrals over $\beta\sigma$ to σ .

(d) It is clear because g is nondecreasing function so either

$$\sigma_1 g(\sigma_2) \leq \sigma_1 g(\sigma_1) \quad \text{or} \quad \sigma_1 g(\sigma_2) \leq \sigma_2 g(\sigma_2).$$

(e) For any $0 < \epsilon < 1$, because of (d) we obtain

$$\sigma_1 g(\sigma_2) = \epsilon \frac{\sigma_1}{\epsilon} g(\sigma_2) \leq \epsilon \left[\frac{\sigma_1}{\epsilon} g\left(\frac{\sigma_1}{\epsilon}\right) + \sigma_2 g(\sigma_2) \right].$$

Applying the right inequality of (0.4) and (b) leads to

$$\sigma_1 g(\sigma_2) \leq \epsilon \left[g_1 G\left(\frac{\sigma_1}{\epsilon}\right) + g_1 G(\sigma_2) \right] \leq \epsilon g_1 \epsilon^{-g_1} G(\sigma_1) + \epsilon g_1 G(\sigma_2).$$

□

The below inequalities will be used to derive the logarithmic energy estimate (4.9) which plays a crucial role in Proposition 3.2.

Lemma 1.2. *For any $\sigma > 0$, let*

$$(1.3) \quad h(\sigma) = \frac{1}{\sigma} \int_0^\sigma g(s) ds,$$

and

$$(1.4) \quad H(\sigma) = \int_0^\sigma h(s) ds.$$

Then we have

$$(a) \quad g_0 h(\sigma) \leq g(\sigma) \leq g_1 h(\sigma).$$

$$(b) \quad g_0 H(\sigma) \leq G(\sigma) \leq g_1 H(\sigma).$$

$$(c) \quad (g_0 - 1)h(\sigma) \leq \sigma h'(\sigma) \leq (g_1 - 1)h(\sigma).$$

$$(d) \quad \frac{1}{g_1} \sigma h(\sigma) \leq H(\sigma) \leq \frac{1}{g_0} \sigma h(\sigma).$$

$$(e) \quad \text{For a constant } \beta > 1,$$

$$\beta^{g_0} H(\sigma) \leq H(\beta\sigma) \leq \beta^{g_1} H(\sigma)$$

Proof. Here we note that h acts like g and H acts like G .

- (a) Dividing (0.4) by σ complete the proof.
- (b) Taking integrals to (a) gives the inequality.
- (c) Since

$$h'(\sigma) = \frac{g(\sigma)}{\sigma} - \frac{G(\sigma)}{\sigma^2},$$

applying (0.4) completes the proof.

- (d) Applying the integration by parts and (a), we yield that

$$\begin{aligned} H(\sigma) &= \sigma h(\sigma) - \int_0^\sigma s h'(s) ds \\ &\leq \sigma h(\sigma) - (g_0 - 1) \int_0^\sigma h(s) ds. \end{aligned}$$

Therefore

$$H(\sigma) \leq \frac{1}{g_0} \sigma h(\sigma).$$

Similarly, we obtain

$$H(\sigma) \geq \frac{1}{g_1} \sigma h(\sigma).$$

- (e) Similar to the proof for (b) on Lemma 1.1.

□

2. THE MAIN LEMMA AND HÖLDER ESTIMATE

The main lemma says that a nonnegative solution u is strictly positive in a subcylinder if u is near to the maximum value in more than a half of cylinder.

Lemma 2.1. *(Main Lemma) Suppose that u is a nonnegative solution in $\Omega_{r,s}^{x_0,t_0}$. Then there exists $R > 0$ such that $Q_{\omega,4R}^{x_0,t_0} \subset \Omega_{r,s}^{x_0,t_0}$. Also there exist positive constants $\theta, \mu \in (0, 1)$, and $\lambda \in (0, 1)$ depending on data such that, if u satisfies*

$$(2.1) \quad \text{meas} \left\{ (x, t) \in Q_{M,2R}^{x_0,t_0} : u(x, t) > \frac{M}{2} \right\} > \frac{1}{2} |Q_{M,2R}^{x_0,t_0}|,$$

then

$$\text{ess inf}_{K_R^{x_0} \times [t_0 - \lambda T_{M,R}, t_0]} u(x, t) \geq \mu M.$$

Proof. The proof of the main lemma will be served in the end of the following section. □

Remark 2.2. *If we work Lemma 2.1 with slightly different assumption that for some constant $\alpha > 0$*

$$\text{meas} \left\{ (x, t) \in Q_{M, 2R}^{x_0, t_0} : u(x, t) > \alpha M \right\} > \frac{1}{2} |Q_{M, 2R}^{x_0, t_0}|,$$

then the constants θ , $\mu \in (0, 1)$, and $\lambda \in (0, 1)$ are depending on data and α .

Lemma 2.3. *Let u be a bounded weak solution of (0.1) with (0.3) in the cylinder $\Omega_{r, s}^{x_0, t_0}$. For R such that $Q_{\omega, 4R}^{x_0, t_0} \subset \Omega_{r, s}^{x_0, t_0}$, suppose that for some constant $\sigma \in (1 - \mu, 1)$*

$$\text{ess osc}_{Q_{\omega, 4R}^{x_0, t_0}} u(x, t) > \sigma \omega.$$

Then

$$(2.2) \quad \text{ess osc}_{K_R^{x_0} \times [t_0 - \lambda T_{\omega, R}, t_0]} u(x, t) \leq \text{ess osc}_{Q_{\omega, 4R}^{x_0, t_0}} u(x, t) + (1 - \sigma - \mu) \omega,$$

where positive constants θ , μ , and λ are from Lemma 2.1.

Proof. Without loss of generality, let $(x_0, t_0) := (0, 0)$. Say $Q := K_R \times [-\lambda T_{\omega, R}, 0]$. Since $Q \subset Q_{\omega, 4R}$, if $\text{ess osc}_{Q_{\omega, 4R}} u(x, t) = 0$, then $\text{ess osc}_Q u(x, t) = 0$. Therefore (2.2) holds.

Suppose that $\text{ess osc}_{Q_{\omega, 4R}} u(x, t) \neq 0$. There are two cases: either

$$(2.3) \quad \text{meas} \left\{ (x, t) \in Q_{\omega, 2R} : u - \text{ess inf}_{Q_{\omega, 4R}} u > \frac{\omega}{2} \right\} > \frac{1}{2} |Q_{\omega, 2R}|$$

or

$$(2.4) \quad \text{meas} \left\{ (x, t) \in Q_{\omega, 2R} : u - \text{ess inf}_{Q_{\omega, 4R}} u \leq \frac{\omega}{2} \right\} > \frac{1}{2} |Q_{\omega, 2R}|.$$

In case (2.3) holds, Lemma 2.1 directly says that there exist $\mu \in (0, 1)$ and a cylinder Q such that

$$\text{ess inf}_Q u - \text{ess inf}_{Q_{\omega, 4R}} u \geq \mu \omega.$$

Therefore we obtain

$$\text{ess osc}_Q u \leq \text{ess sup}_{Q_{\omega, 4R}} u - \text{ess inf}_{Q_{\omega, 4R}} u - \mu \omega$$

that leads to the conclusion.

In case (2.4) holds, the inequality

$$u(x, t) - \text{ess inf}_{Q_{\omega, 4R}} u(x, t) \leq \omega/2$$

implies

$$(1 - \sigma) \omega + \text{ess sup}_{Q_{\omega, 4R}} u(x, t) - u(x, t) \geq \omega/2.$$

By applying Lemma 2.1 to the nonnegative weak solution

$$(1 - \sigma)\omega + \operatorname{ess\,sup}_{Q_{\omega,4R}} u(x, t) - u(x, t),$$

it follows

$$\operatorname{ess\,sup}_Q u \leq \operatorname{ess\,sup}_{Q_{\omega,4R}} u(x, t) + (1 - \sigma - \mu)\omega.$$

We finish our proof noting that $\mu + \sigma > 1$. \square

Now based on Lemma 2.3, we can fit in a sequence of shrinking and nested cylinders.

Lemma 2.4. *Let u be a bounded weak solution of (0.1) with (0.3) in $\Omega_{r,s}^{x_0,t_0}$. Then there exist a constant $\delta \in (0, 1)$ and a family of shrinking and nested cylinders $\{Q_n\}_{n=0}^\infty$ such that*

$$(2.5) \quad \operatorname{ess\,osc}_{Q_{n+1}} u(x, t) \leq \delta^n \delta \operatorname{ess\,osc}_{Q_n} u(x, t).$$

Proof. Without loss of generality, let $(x_0, t_0) := (0, 0)$. Choose a positive constant ω_0 such that $\omega_0 \geq \operatorname{ess\,osc}_{\Omega_{r,s}} u(x, t)$.

First, we fix a positive constant

$$(2.6) \quad \rho_0 := \min \left\{ r, \frac{\omega_0}{G^{-1}(\theta\omega_0^2 s^{-1})} \right\}$$

that is driven for cylinder inclusion

$$Q_0 := K_{\rho_0} \times [-\theta\omega_0^2 G \left(\frac{\omega_0}{\rho_0} \right)^{-1}, 0] \subset \Omega_{r,s}.$$

Then clearly, $\operatorname{ess\,osc}_{Q_0} u(x, t) \leq \omega_0$.

Now we introduce two constants $\delta \in (0, 1)$ and $\epsilon \in (0, 1)$ and set

$$\omega_n := \delta^n \omega_0, \quad \rho_n := \epsilon^n \rho_0.$$

Moreover, we build a shrinking and nested sequence of cylinders about $(0, 0)$ such that

$$Q_n := K_{\rho_n} \times [-\theta\omega_n^2 G \left(\frac{\omega_n}{\rho_n} \right)^{-1}, 0]$$

where

$$(2.7) \quad 4\epsilon = \min \left\{ \delta, 4\delta^{\frac{g_0-2}{g_0}}, \lambda^{\frac{1}{g_0}} \delta^{\frac{g_0-2}{g_0}} \right\}$$

and

$$(2.8) \quad \delta = \max \{ \sigma, 1 - (\sigma + \mu - 1) \}$$

with constants $\lambda \in (0, 1)$ and $\mu \in (0, 1)$ from Lemma 2.1 and $\sigma \in (1 - \mu, 1)$ from Lemma 2.3.

The choice of ϵ in (2.7) is made to satisfy

$$(2.9a) \quad Q_{n+1} \subset Q_n,$$

$$(2.9b) \quad Q_{n+1} \subset K_{\rho_n/4} \times [-\lambda\theta\omega_n^2 G\left(\frac{\omega_n}{\rho_n/4}\right)^{-1}, 0].$$

Under $4\epsilon \leq \delta$, owing to the right hand side of (1.2) from Lemma 1.1, (2.9) is guaranteed by two inequalities

$$\delta^2 \left(\frac{\epsilon}{\delta}\right)^{g_0} \leq 1, \quad \delta^2 \left(\frac{4\epsilon}{\delta}\right)^{g_0} \leq \lambda$$

that directly generates (2.7).

To determine a constant δ , there are two cases to consider: either, for a constant $\sigma \in (1 - \mu, 1)$,

$$\begin{aligned} \operatorname{ess\,osc}_{Q_{n+1}} u(x, t) &\leq \sigma \operatorname{ess\,osc}_{Q_n} u(x, t), \\ \operatorname{ess\,osc}_{Q_{n+1}} u(x, t) &\geq \sigma \operatorname{ess\,osc}_{Q_n} u(x, t). \end{aligned}$$

Due to a cylinder relationship (2.9b), in the second case, we apply Lemma 2.3. Therefore we conclude (2.8). \square

Here we define the length of time using the function G such that

$$(2.10) \quad |t - s|_G = \frac{\|u\|_\infty}{G^{-1}(\theta\|u\|_{\infty, \Omega_T}^2/|t - s|)}$$

which is basically from (2.6). By using the time length defined as in (2.10), we define the length of two sets such that

$$\operatorname{dist}(\mathcal{K}_1; \mathcal{K}_2) := \inf_{(x,t) \in \mathcal{K}_1, (y,s) \in \mathcal{K}_2} (|x - y| + |t - s|_G).$$

Because of generalized function G , it is natural to obtain a modulus of continuity with a presence of G . Then later, with an extra assumption, we are able to derive a Hölder estimate written in terms of exact powers.

Theorem 2.5. *Let u be a bounded weak solution of (0.1) with (0.3) in $\Omega_{r,s}^{x_0,t_0}$. Then $(x, t) \rightarrow u(x, t)$ has modulus of continuity. Moreover, there exists constant γ and $\alpha \in (0, 1)$ depending only upon the data such that, for any two distinct points (x_1, t_1) and (x_2, t_2) in any set Ω' which is subset of $\Omega_{r,s}^{x_0,t_0}$ strictly away from $\partial_p \Omega_{r,s}^{x_0,t_0}$,*

$$|u(x_1, t_1) - u(x_2, t_2)| \leq \gamma \|u\|_\infty \left(\frac{|x_1 - x_2| + |t_1 - t_2|_G}{\operatorname{dist}(\Omega'; \partial_p \Omega_{r,s}^{x_0,t_0})} \right)^\alpha.$$

Proof. In the cylinder Ω' , we construct a sequence $\{Q_n\}_{n=0}^\infty$ of cylinders as in Lemma 2.4. Set a sequence $\{\omega_n\}_{n=0}^\infty$ such that $\omega_n \geq \text{ess osc}_{Q_n} u(x, t)$.

Consider $r > 0$ such that

$$(2.11) \quad \rho_{n+1} < r \leq \rho_n \quad \text{for some } n.$$

Also consider $s > 0$ such that

$$(2.12) \quad \theta \omega_{m+1}^2 G \left(\frac{\omega_{m+1}}{\rho_{m+1}} \right)^{-1} < s \leq \theta \omega_m^2 G \left(\frac{\omega_m}{\rho_m} \right)^{-1} \quad \text{for some } m.$$

As a result, we obtain that

$$\text{ess osc}_{\Omega_{r,s}} u(x, t) \leq \max\{\omega_n, \omega_m\}.$$

From the left hand side inequality of (2.11), we derive

$$\frac{r}{\rho_0} > \epsilon^{n+1} = (\delta^{\log_\delta \epsilon})^{n+1}$$

which implies by setting $\alpha_1 = \log_\epsilon \delta$

$$\omega_n = \delta^n \omega_0 < \delta^{-1} \omega_0 \left(\frac{r}{\rho_0} \right)^{\alpha_1}.$$

On the other hand, the left hand side of (2.12) delivers that

$$\frac{s}{\theta \omega_0^2 G(\omega_0/\rho_0)^{-1}} > \delta^{(m+1)\{(2-g_0)+g_0 \log_\delta \epsilon\}}$$

due to $\epsilon < \delta$. Also the choice of epsilon that $\epsilon < \delta^{(g_0-2)/g_0}$ implies that

$$(2 - g_0) + g_0 \log_\delta \epsilon < 0.$$

Hence by letting $\alpha_2 = \log_{\delta^{2-g_0} \epsilon^{g_0}} \delta$, we have

$$\omega_m \leq \left(\delta^{\alpha_2} \frac{s}{\theta \omega_0^2 G(\omega_0/\rho_0)^{-1}} \right)^{\alpha_2}.$$

Therefore, for some $\gamma > 0$,

$$\text{ess}_{\Omega'} \text{osc } u(x, t) \leq \gamma \left[\left(\frac{r}{\rho_0} \right)^{\alpha_1} + \left(\frac{s}{\theta \omega_0^2 G(\omega_0/\rho_0)^{-1}} \right)^{\alpha_2} \right].$$

Clearly $\omega_0 \leq 2\|u\|_\infty$ and $\rho_0 \geq \text{dist}(\Omega'; \partial_p \Omega_{r,s}^{x_0, t_0})$ which implies our conclusion. \square

Corollary 2.6. *Let u be a bounded weak solution of (0.1) with (0.3) in $\Omega_{r,s}^{x_0, t_0}$. Then $(x, t) \rightarrow u(x, t)$ is locally Hölder continuous. Moreover, there exists positive constants γ, β and $\alpha \in (0, 1)$ depending only upon*

the data such that, for any two distinct points $(x_1, t_1), (x_2, t_2)$ in any set Ω' which is subset of $\Omega_{r,s}^{x_0, t_0}$ strictly away from $\partial_p \Omega_{r,s}^{x_0, t_0}$,

$$(2.13) \quad |u(x_1, t_1) - u(x_2, t_2)| \leq \gamma \|u\|_{\infty, \Omega_T} \left(\frac{|x_1 - x_2| + \beta \theta^{-\frac{1}{g_1}} \|u\|_{\infty}^{\frac{g_1-2}{g_1}} |t_1 - t_2|^{\frac{1}{g_1}}}{\tilde{\text{dist}}(\Omega'; \partial_p \Omega_{r,s}^{x_0, t_0})} \right)^{\alpha}$$

where

$$(2.14) \quad \tilde{\text{dist}}(\Omega', \partial_p \Omega_{r,s}^{x_0, t_0}) = \inf_{(x,t) \in \Omega', (y,s) \in \partial_p \Omega_{r,s}^{x_0, t_0}} \left[|x - y| + \beta \theta^{-\frac{1}{g_0}} \|u\|_{\infty}^{\frac{g_0-2}{g_0}} |t - s|^{\frac{1}{g_0}} \right].$$

Proof. For simplicity, define a constant $\beta > 0$ such that $G(1/\beta) = 1$. Besides choosing ρ_0 from (2.6) in Lemma 2.4, we add an extra condition that

$$(2.15) \quad \rho_0 \leq \beta \omega_0.$$

Then we derive that

$$(2.16) \quad \theta \omega_0^2 \leq s \left(\frac{\beta \omega_0}{\rho_0} \right)^{g_0} \quad \text{implies} \quad s \geq \theta \omega_0^2 G \left(\frac{\omega_0}{\rho_0} \right)^{-1}.$$

Therefore we derive a sort of power distance for ρ_0 such that

$$\rho_0 \leq \beta \left(\frac{\omega_0^{g_0-2}}{\theta} \right)^{\frac{1}{g_0}} s^{\frac{1}{g_0}}$$

from which we obtain (2.14).

The condition that $\epsilon < \delta$ from Theorem 2.5 with (2.15) clearly implies that $\rho_n < \alpha \omega_n$ for all $n = 0, 1, 2, \dots$. Therefore from the left hand side of the inequality (2.12), we have that

$$\theta \omega_{m+1}^2 \left(\frac{\beta \omega_{m+1}}{\rho_{m+1}} \right)^{-g_1} < s$$

which is equivalent to

$$\delta^{(m+1)\{g_1 \log_{\delta} \epsilon + (2-g_1)\}} < \frac{\beta^{g_1} \omega_0^{g_1-2} s}{\theta \rho_0^{g_1}}.$$

Here we note that

$$g_1 \log_{\delta} \epsilon + (2 - g_1) < 0$$

because of (2.7). Now by letting

$$\alpha_2 := \log_{\epsilon^{g_1} \delta^{2-g_1}} \delta,$$

we obtain that

$$\omega^m \leq \delta^{-1} \omega_0 \left(\frac{\beta^{g_1} \theta^{-1} \omega_0^{g_1-2} s}{\rho_0^{g_1}} \right)^{\alpha_2}.$$

□

Remark 2.7. *From Corollary 2.6, we can observe that the initial scaling is determined by the power g_0 . Then g_1 plays its role later. Especially, when $1 < g_0 < 2$, the extinction of a solution may occur in finite time. Therefore a cylinder can not be too long and initial scaling with g_0 makes sense.*

3. PROOF OF THE MAIN LEMMA

Throughout this section, let u to be a bounded nonnegative weak solution of (0.1) with (0.3). The proof of Lemma 2.1 is composed with four steps under the assumption that u is large at least half of a cylinder $Q_{\omega, 2R}$. Then Proposition 3.1 implies that a spatial cube at some fixed time level is found on which u is away from its minimum (zero value) on arbitrary fraction of the spatial cube. From the spatial cube, positive information spread in both later time and over the space variables with time limitations (Proposition 3.2 & Proposition 3.3). Controlling the positive quantity $\theta > 0$ on $T_{k, \rho}$ is key to overcome those time restrictions. Once we have a subcylinder centered at $(0, 0)$ in $Q_{\omega, 4R}$ with arbitrary fraction of the subcylinder, we finally apply modified De Giorgi iteration (Proposition 3.4) to obtain strictly positive infimum of u in a smaller cylinder around $(0, 0)$. We can carry analogous proof when u is away from its maximum (u is close to its minimum) at least half of cylinder.

Parts of proof for the following Proposition comes from Proposition 3.7 in [12], Lemmata III.7.1, IV.10.1 in [8], and concerning the equation (4.2) on page 35 in [22].

Proposition 3.1. *For a given constant $k > 0$ and $\rho > 0$, suppose that u is a nonnegative weak solution on $Q_{k, 2\rho}$ satisfying*

$$(3.1) \quad \text{meas} \left\{ (x, t) \in Q_{k, \rho} : u(x, t) > \frac{k}{2} \right\} \geq \frac{1}{2} |Q_{k, \rho}|$$

Then for any $\nu_1 \in (0, 1)$ and $\delta_1 \in (0, 1/2]$, there exist $y \in K_\rho$, $-\tau_1 \in [-T_{k, \rho}, -T_{k, \rho}/16]$ and $\eta \in (0, 1)$ such that $K_{\eta\rho}^y \subset K_\rho$ and

$$\text{meas} \{ x \in K_{\eta\rho}^y : u(x, -\tau_1) < \delta_1 k \} < (1 - \nu_1) |K_{\eta\rho}^y|.$$

Proof. We apply the local energy estimate (4.1) with a piecewise linear cutoff function

$$\zeta = \begin{cases} 1 & \text{inside } Q_{k,\rho} \\ 0 & \text{on the parabolic boundary of } Q_{k,2\rho} \end{cases}$$

with

$$|D\zeta| \leq \frac{1}{\rho}, \quad \zeta_t \leq \frac{1}{(2^{g_0} - 1)\theta k^2} G\left(\frac{k}{\rho}\right).$$

It follows that

$$\begin{aligned} & \int_{-T_{k,2\rho}}^0 \int_{K_{2\rho}} G(|D(u - k/2)_-|) G^{r-1}\left(\frac{\zeta(u - k/2)_-}{\rho}\right) (u - k/2)_-^s \zeta^q dx dt \\ & \leq \gamma_1 \int_{-T_{k,2\rho}}^0 \int_{K_{2\rho}} G^{r-1}\left(\frac{\zeta(u - k/2)_-}{\rho}\right) (u - k/2)_-^{s+2} \zeta^{q-1} \zeta_t dx dt \\ & \quad + \gamma_2 \int_{-T_{k,2\rho}}^0 \int_{K_{2\rho}} G^r\left(\frac{\zeta(u - k/2)_-}{\rho}\right) (u - k/2)_-^s \zeta^{q-1-2g_1} dx dt, \end{aligned}$$

for some constants γ_1 and γ_2 . Note that

$$(u - k/2)_- = \max\{0, (k/2 - u)\} \leq k/2.$$

Moreover, constants r and s are chosen such that the map $\sigma \mapsto G^{r-1}(\sigma)\sigma^s$ is nonincreasing and the maps $\sigma \mapsto G^{r-1}(\sigma)\sigma^{s+2}$ and $\sigma \mapsto G^r(\sigma)\sigma^s$ are increasing.

Therefore we obtain that

$$\begin{aligned} & \int_{-T_{k,2\rho}}^0 \int_{K_{2\rho}} G(|D(u - k/2)_-|) \zeta^q dx dt \\ & \leq \left\{ \gamma_1 \left(\frac{k}{2}\right)^2 \frac{G(k/\rho)}{(2^{g_0}-1)\theta k^2} + \gamma_2 G\left(\frac{k/2}{\rho}\right) \right\} |K_{2\rho} \times [-T_{k,2\rho}, 0]| \\ & \leq \tilde{\gamma} G\left(\frac{k}{\rho}\right) |K_{2\rho} \times [-T_{k,2\rho}, 0]|. \end{aligned}$$

Then Jensen's inequality provides

$$(3.2) \quad \int_{-T_{k,\rho}}^0 \int_{K_\rho} |D(u - k/2)_-| dx dt \leq \frac{\gamma}{\rho} |K_\rho \times [-T_{k,\rho}, 0]|$$

for some constant γ .

Now we say that there exists $\tau_1 \in [-T_{k,\rho}, -T_{k,\rho}/16]$ satisfying both

$$(3.3a) \quad \int_{K_\rho} |D(u - k/2)_-|(x, -\tau_1) dx \leq \frac{16\gamma}{\rho} |K_\rho|,$$

$$(3.3b) \quad |\{u(x, -\tau_1) \geq k/2\} \cap K_\rho| \geq \frac{5}{8} |K_\rho|.$$

If the inequality (3.3a) fails in the time set with more than $T_{k,\rho}/16$ measure, then clearly it produces contradiction to the inequality (3.2). If (3.3b) fails in the set with more than $T_{k,\rho}/8$ measure set, then we derive

$$\begin{aligned} \text{meas} \{Q_{k,\rho} : u(x, t) \geq k/2\} &= |Q_{k,\rho}| - \text{meas} \{Q_{k,\rho} : u(x, t) < k/2\} \\ &\leq \left(1 - \frac{5}{2 \cdot 8} \left(1 - \frac{1}{8}\right)\right) |Q_{k,\rho}| \\ &= \frac{29}{64} |Q_{k,\rho}| < \frac{1}{2} |Q_{k,\rho}| \end{aligned}$$

that contradicts to our assumption (3.1). Therefore in the set $[-T_{k,\rho}, 0]$, the inequality (3.3a) holds in more than set with measure $15T_{k,\rho}/16$ and the inequality (3.3b) is true for more than $7T_{k,\rho}/8$. Thus, there exists $\tau_1 \in [-T_{k,\rho}/16, 0]$ where (3.3) hold. Our conclusion is made after applying Lemma 4.3 which is quoted from [9]. \square

Proposition 3.2 is similar to Lemmata III.4.1, III.7.2, IV.10.2 from [8]. If $g_0 > 2$, then the next proposition can be replaced by Corollary 3.4 from [12] which does not involve the logarithmic energy estimate.

Proposition 3.2. *Let constants $\nu \in (0, 1)$, $k > 0$, and $\rho > 0$ be given. Then, for any $\epsilon \in (0, 1)$, there exists a nonnegative integer $j = j(\nu, N, g_1)$ such that, if*

$$(3.4) \quad \text{meas} \{x \in K_\rho^y : u(x, -\tau) < k\} < (1 - \nu) |K_\rho^y|$$

for some

$$(3.5) \quad \begin{cases} \tau \leq k^2 G\left(\frac{k}{\rho}\right)^{-1} & \text{if } g_0 \geq 2, \\ \tau \leq (2^{-j}k)^2 G\left(\frac{2^{-j}k}{\rho}\right)^{-1}, & \text{if } g_1 \leq 2, \end{cases}$$

then

$$\text{meas} \{x \in K_\rho^y : u(x, -t) < 2^{-j}k\} < (1 - (1 - \epsilon)\nu) |K_\rho^y|$$

for any $-t \in (-\tau, 0]$.

Proof. Here we apply the logarithmic energy estimate (4.9) on the cylinder $K_\rho^y \times [-\tau, 0]$ with a piecewise linear cutoff function

$$\zeta = \begin{cases} 1 & \text{inside } K_{(1-\sigma)\rho}^y \times [-\tau, 0] \\ 0 & \text{on the lateral boundaries of } K_\rho^y \times [-\tau, 0]. \end{cases}$$

with

$$|D\zeta| \leq \frac{1}{\sigma\rho}, \quad \zeta_t = 0$$

where $\sigma \in (0, 1)$ will be determined later. For a positive constant γ and $q = g_1$, we have

$$(3.6) \quad \begin{aligned} \int_{K_\rho^y \times \{0\}} H(\Psi^2) \zeta^{g_1} dx &\leq \int_{K_\rho^y \times \{-\tau\}} H(\Psi^2) \zeta^{g_1} dx \\ &+ \gamma \int_{-\tau}^0 \int_{K_\rho^y} h(\Psi^2) |\Psi| |\Psi'|^2 G\left(\frac{|D\zeta|}{|\Psi'|}\right) dx dt, \end{aligned}$$

where h and H are defined in Lemma 1.2.

Let $\delta = 2^{-j}$ where j to be chosen large enough. We recall

$$\Psi = \ln^+ \left[\frac{k}{(1+\delta)k - (u-k)_-} \right], \quad \Psi' = \frac{-1}{(1+\delta)k - (u-k)_-}.$$

Since $0 \leq (u-k)_- \leq k$, we have

$$\Psi \leq \ln^+ \delta^{-1} = j \ln 2, \quad \frac{1}{(1+\delta)k} \leq |\Psi'| \leq \frac{1}{\delta k}.$$

Moreover, in the set $\{u < (1-2^{-j})k\}$, we obtain a lower bound

$$\Psi \geq \ln^+(2\delta)^{-1} = (j-1) \ln 2.$$

The left hand side of the inequality (3.6) is lower bounded

$$\begin{aligned} &\int_{K_\rho^y \times \{-t\}} H(\Psi^2) \zeta^{g_1} dx \\ &\geq H((j-1)^2 (\ln 2)^2) \text{meas} \left\{ x \in K_{(1-\sigma)\rho}^y : u(x, -t) < \delta k \right\}. \end{aligned}$$

Due to (3.4), the first integral term on the right hand side of (3.6) is bounded by

$$\int_{K_\rho^y \times \{-\tau\}} H(\Psi^2) \zeta^{g_1} dx \leq H(j^2 (\ln 2)^2) (1-\nu) |K_\rho^y|.$$

From the upper and lower bounds of $|\Psi'|$, we observe that

$$\frac{1/|\Psi'|}{\delta k} \geq 1, \quad \frac{1/|\Psi'|}{(1+\delta)k} \leq 1,$$

which provide

$$\begin{aligned} G\left(\frac{|D\zeta|}{|\Psi'|}\right) &\leq \left(\frac{1/|\Psi'|}{\delta k}\right)^{g_1} G(\delta k|D\zeta|), \\ G\left(\frac{|D\zeta|}{|\Psi'|}\right) &\leq \left(\frac{1/|\Psi'|}{(1+\delta)k}\right)^{g_0} G((1+\delta)k|D\zeta|). \end{aligned}$$

Hence, for $g_1 \leq 2$, it follows that

$$\tau|\Psi'|^2 G\left(\frac{|D\zeta|}{|\Psi'|}\right) \leq \left(\frac{1/|\Psi'|}{\delta k}\right)^{g_1-2} \sigma^{-g_1} \leq \sigma^{-g_1}$$

and, for $g_0 \geq 2$,

$$\tau|\Psi'|^2 G\left(\frac{|D\zeta|}{|\Psi'|}\right) \leq \left(\frac{1/|\Psi'|}{(1+\delta)k}\right)^{g_0-2} (1+\delta)^{g_1-2} \sigma^{-g_1} \leq 2^{g_1-2} \sigma^{-g_1}.$$

Therefore we obtain the upper bound of the second integral on the right hand side of (3.6) such that

$$\begin{aligned} &\int_{-\tau}^{-t} \int_{K_\rho^y} h(\Psi^2)|\Psi||\Psi'|^2 G\left(\frac{|D\zeta|}{|\Psi'|}\right) dx dt \\ &\leq \begin{cases} h(j^2(\ln 2)^2) j(\ln 2) \sigma^{-g_1} \tau |K_\rho^y| & \text{if } g_1 \leq 2, \\ h(j^2(\ln 2)^2) j(\ln 2) 2^{g_1-2} \sigma^{-g_1} \tau |K_\rho^y| & \text{if } g_0 \geq 2. \end{cases} \end{aligned}$$

because of (1.3), we note that

$$h(j^2(\ln 2)^2) j \ln 2 \leq g_1 \frac{H(j^2(\ln 2)^2)}{j \ln 2}.$$

For any $-t \in (-\tau, 0]$, we yield

$$\begin{aligned} &\text{meas} \{x \in K_\rho^y : u(x, -t) < \delta k\} \\ &\leq \left[(1-\nu) \frac{H(j^2(\ln 2)^2)}{H((j-1)^2(\ln 2)^2)} + \gamma \frac{H(j^2(\ln 2)^2)}{H((j-1)^2(\ln 2)^2)} \frac{1}{j(\ln 2)^{\sigma g_1}} + N\sigma \right] |K_\rho^y| \end{aligned}$$

where γ is a constant depending on the data. For brevity, say

$$H' = \frac{H(j^2(\ln 2)^2)}{H((j-1)^2(\ln 2)^2)}.$$

For a fixed constant $\epsilon \in (0, 1)$, we select a positive integer j large enough and $\sigma \in (0, 1)$ so that

$$(3.7a) \quad H' \leq 1 + \epsilon\nu,$$

$$(3.7b) \quad \gamma \frac{H'}{j(\ln 2)\sigma^{g_1}} \leq \frac{\epsilon\nu^2}{2},$$

$$(3.7c) \quad N\sigma \leq \frac{\epsilon\nu^2}{4}.$$

Then inequalities (3.7) yield our conclusion. Therefore we complete the proof by going back to (3.7) and finding j and σ . From inequalities (3.7c), we first choose

$$\sigma = \frac{\epsilon\nu^2}{2N}.$$

Assuming (3.7a), the inequality (3.7b) implies that

$$j \ln 2 \geq (1 + \epsilon\nu) \frac{4g_1\gamma}{\nu^2} \left[1 + \left(\frac{2N}{\epsilon\nu^2} \right)^{g_1} \right]$$

so that, for a constant γ depending on the data,

$$j \geq \gamma (\epsilon\nu^2)^{-1-g_1}.$$

Because of Lemma 1.2 and (3.7a), it follows that

$$\left(\frac{j}{j-1} \right)^{g_1} \leq 1 + \epsilon\nu$$

which provides

$$j \geq 1 + (\epsilon\nu)^{-1/g_1}.$$

Hence we choose a positive integer j large enough such that

$$j \geq \max \left\{ \gamma (\epsilon\nu^2)^{-1-g_1}, 1 + (\epsilon\nu)^{-1/g_1} \right\}.$$

□

The following proposition says that positive information in K_ρ for all time expands to $K_{2\rho}$ for comparable times. Spreading positivity is natural when $g_1 < 2$ because the modulus of parabolicity is dominating when $|Du| \rightarrow 0$ but when $g_1 > 2$, enough time length is required for spreading positivity over the spatial cube. The Proposition 3.3 is analogous to Lemma 3.5 from [12], Theorem 1.1 from [11], Proposition 6.1 from [10], and Lemma IV.11.1 from [8].

Proposition 3.3. *For given $k > 0$, $\rho > 0$, $y \in K_\rho$, $\eta \in (0, 1)$, and $\alpha \in (0, 1)$, suppose that $K_{\eta\rho}^y \subset K_\rho$. Then for any $\nu \in (0, 1)$, there exists a positive integer $j^* = j^*(N, \alpha, g_1, \eta, \nu)$ such that, if*

$$(3.8) \quad \text{meas} \{x \in K_{\eta\rho}^y : u(x, -t) < k\} < (1 - \alpha) |K_{\eta\rho}^y|$$

for all $-t \in (-2\tau, 0]$ where

$$(3.9) \quad \begin{cases} \tau \geq k^2 G\left(\frac{k}{\rho}\right)^{-1} & \text{if } g_1 \leq 2, \\ \tau \geq (2^{-j^*} k)^2 G\left(\frac{2^{-j^*} k}{\rho}\right)^{-1} & \text{if } g_0 \geq 2, \end{cases}$$

then

$$(3.10) \quad \text{meas} \{(x, t) \in K_\rho \times [-\tau, 0] : u(x, t) < 2^{-j^*} k\} < \nu |K_\rho \times [-\tau, 0]|.$$

Proof. Let $k_j = 2^{-j} k$ for $j = 0, 1, 2, \dots, j^*$ with j^* to be determined later. For simplicity, denote

$$A_j = \{(x, t) \in K_\rho \times [-\tau, 0] : u(x, t) < k_j\}$$

We work with a piecewise linear cutoff function that

$$\zeta = \begin{cases} 1 & \text{inside of } K_\rho \times [-\tau, 0] \\ 0 & \text{on the parabolic boundary of } K_{2\rho} \times [-2\tau, 0] \end{cases}$$

with

$$|D\zeta| \leq \frac{1}{\rho}, \quad \zeta_t \leq \frac{1}{\tau}.$$

The local energy estimate (4.1) (by ignoring the first term on the left hand side) provides

$$(3.11) \quad \begin{aligned} & \int_{-2\tau}^0 \int_{K_{2\rho}} G(|D(u - k_j)_-|) G^{r-1} \left(\frac{\zeta(u - k_j)_-}{\rho} \right) (u - k_j)_-^s \zeta^q dx dt \\ & \leq \gamma_1 \int_{-2\tau}^0 \int_{K_{2\rho}} G^{r-1} \left(\frac{\zeta(u - k_j)_-}{\rho} \right) (u - k_j)_-^{s+2} \zeta^{q-1} \zeta_t dx dt \\ & \quad + \gamma_2 \int_{-2\tau}^0 \int_{K_{2\rho}} G^r \left(\frac{\zeta(u - k_j)_-}{\rho} \right) (u - k_j)_-^s \zeta^{q-1-2g_1} dx dt. \end{aligned}$$

We first note that for any positive constants $a < b$, it follows that if $g_1 \leq 2$, then

$$\left(\frac{a}{b}\right)^2 \frac{G(b)}{G(a)} \leq \left(\frac{a}{b}\right)^{2-g_1} \leq 1$$

and if $g_0 \geq 2$, then

$$\left(\frac{a}{b}\right)^2 \frac{G(b)}{G(a)} \geq \left(\frac{a}{b}\right)^{2-g_0} \geq 1.$$

Therefore, for any $0 < a < b$, we say that

$$a^2 G(a)^{-1} \leq b^2 G(b)^{-1} \text{ if } g_1 \leq 2,$$

$$a^2 G(a)^{-1} \geq b^2 G(b)^{-1} \text{ if } g_0 \geq 2.$$

Hence, this property of G , we obtain that

$$k_j^2 G\left(\frac{k_j}{\rho}\right)^{-1} \leq k^2 G\left(\frac{k}{\rho}\right)^{-1} \text{ if } g_1 \leq 2,$$

$$k_j^2 G\left(\frac{k_j}{\rho}\right)^{-1} \leq k_{j^*}^2 G\left(\frac{k_{j^*}}{\rho}\right)^{-1} \text{ if } g_0 \geq 2,$$

that provide the proper choice of time τ so that, for any $j = 0, \dots, j^*$,

$$\tau \geq k_j^2 G\left(\frac{k_j}{\rho}\right).$$

Therefore, the integral estimate (3.11) is reduced to

$$(3.12) \quad \int_{-\tau}^0 \int_{K_\rho} G(|D(u - k_j)_-|) \, dx \, dt \leq \gamma G\left(\frac{k_j}{\rho}\right) |K_{2\rho} \times [-2\tau, 0]|.$$

Owing to the assumption (3.8), we apply a Poincare type inequality, Corollary 4.5. For any $-t \in [-\tau, 0]$, it follows that

$$\begin{aligned} & (k_j - k_{j+1}) \text{ meas } \{x \in K_\rho : u(x, -t) < k_{j+1}\} \\ & \leq \frac{\rho^{N+1}}{\alpha(\eta\rho)^N} \int_{K_\rho \cap \{k_{j+1} \leq u < k_j\}} |D(u - k_j)_-| \, dx. \end{aligned}$$

Note $k_j - k_{j+1} = k_{j+1}$. After taking integral over the time variable from $-\tau$ to 0, we obtain

$$(3.13) \quad \frac{k_{j+1}}{\rho} |A_{j+1}| \leq \frac{1}{\alpha\eta^N} \iint_{A_j \setminus A_{j+1}} |D(u - k_j)_-| \, dx \, dt.$$

After dividing (3.13) by $|A_j \setminus A_{j+1}|$, apply Jensen's inequality which implies

$$(3.14) \quad G\left(\frac{|A_{j+1}|}{|A_j \setminus A_{j+1}|} \frac{k_{j+1}}{\rho}\right) \leq \frac{1}{\alpha\eta^N |A_j \setminus A_{j+1}|} \iint_{A_j \setminus A_{j+1}} G(|D(u - k_j)_-|) \, dx \, dt.$$

Because of (3.12), the inequality (3.14) generates

$$(3.15) \quad G\left(\frac{|A_{j+1}|}{|A_j \setminus A_{j+1}|} \frac{k_{j+1}}{\rho}\right) \leq \frac{\gamma 2^{N+1} |K_\rho \times [-\tau, 0]|}{\alpha\eta^N |A_j \setminus A_{j+1}|} G\left(\frac{k_j}{\rho}\right).$$

Denote $\Omega_\tau := K_\rho \times [-\tau, 0]$. There are two cases to consider for any j : either

$$\begin{aligned} |A_{j+1}| &\leq |A_j \setminus A_{j+1}|, \\ |A_{j+1}| &> |A_j \setminus A_{j+1}|. \end{aligned}$$

First, if $|A_{j+1}| \leq |A_j \setminus A_{j+1}|$, then we observe that

$$\left(\frac{|A_{j+1}|}{|A_j \setminus A_{j+1}|} \right)^{g_1} 2^{-g_1} G\left(\frac{k_j}{\rho}\right) \leq G\left(\frac{|A_{j+1}|}{|A_j \setminus A_{j+1}|} \frac{k_{j+1}}{\rho}\right).$$

The inequality (3.15) gives

$$(3.16) \quad \left(\frac{|A_{j+1}|}{|A_j \setminus A_{j+1}|} \right)^{g_1} 2^{-g_1} \leq \frac{\gamma 2^{N+1}}{\alpha \eta^N} \frac{|\Omega_\tau|}{|A_j \setminus A_{j+1}|}$$

which is

$$(3.17) \quad \left(\frac{|A_{j+1}|}{|\Omega_\tau|} \right)^{\frac{g_1}{g_1-1}} \leq \left(\frac{\gamma 2^{N+1+g_1}}{\alpha \eta^N} \right)^{\frac{1}{1-g_1}} \frac{|A_j \setminus A_{j+1}|}{|\Omega_\tau|}.$$

Second, if $|A_{j+1}| > |A_j \setminus A_{j+1}|$ holds, then we have

$$\left(\frac{|A_{j+1}|}{|A_j \setminus A_{j+1}|} \right)^{g_0} 2^{-g_1} G\left(\frac{k_j}{\rho}\right) \leq G\left(\frac{|A_{j+1}|}{|A_j \setminus A_{j+1}|} \frac{k_{j+1}}{\rho}\right).$$

Therefore, (3.15) generates

$$(3.18) \quad \left(\frac{|A_{j+1}|}{|\Omega_\tau|} \right)^{\frac{g_0}{g_0-1}} \leq \left(\frac{2^{N+1+g_1} \gamma}{\alpha \eta^N} \right)^{\frac{1}{g_0-1}} \frac{|A_j \setminus A_{j+1}|}{|\Omega_\tau|}.$$

Next we take sum for $j = 0, \dots, j^* - 1$ of the inequality (3.13). Note that $|A_{j^*}| \leq |A_{j+1}|$ for all $j = 0, \dots, j^* - 1$. Owing to (??), (3.17), and (3.18), it follows that

$$(3.19) \quad \begin{aligned} &j^* \min \left\{ \left(\frac{|A_{j^*}|}{|\Omega_\tau|} \right)^{\frac{g_0}{g_0-1}}, \left(\frac{|A_{j^*}|}{|\Omega_\tau|} \right)^{\frac{g_1}{g_1-1}} \right\} \\ &\leq \max \left\{ (\beta)^{\frac{1}{g_0-1}}, (\beta)^{\frac{1}{g_1-1}} \right\} \end{aligned}$$

where

$$\beta = \frac{\gamma 2^{N+1+g_1}}{\alpha \eta^N}.$$

Since $g_0 \leq g_1$, for $\beta \geq 1$,

$$\max \left\{ (\beta)^{\frac{1}{g_0-1}}, (\beta)^{\frac{1}{g_1-1}} \right\} \leq (\beta)^{\frac{1}{g_0-1}}.$$

From the inequality

$$j^* \left(\frac{|A_{j^*}|}{|\Omega_\tau|} \right)^{\frac{g_0}{g_0-1}} \leq (\beta)^{\frac{1}{g_0-1}},$$

we reach to our conclusion (3.10) by choosing j^* such that

$$j^* \geq \nu^{\frac{g_0}{1-g_0}} \beta^{\frac{1}{g_0-1}}$$

for any $\nu \in (0, 1)$. Moreover, to obtain same conclusion from the inequality

$$j^* \left(\frac{|A_{j^*}|}{|\Omega_\tau|} \right)^{\frac{g_1}{g_1-1}} \leq (\beta)^{\frac{1}{g_0-1}},$$

the constant j^* has to be chosen so that

$$j^* \geq \nu^{\frac{g_1}{1-g_1}} \beta^{\frac{1}{g_0-1}}.$$

Therefore we complete the proof by setting

$$j^* \geq \max \left\{ \nu^{\frac{g_0}{1-g_0}} \beta^{\frac{1}{g_0-1}}, \nu^{\frac{g_1}{1-g_1}} \beta^{\frac{1}{g_0-1}} \right\}.$$

□

The following proposition is modified DeGiorgi iteration with generalized structure conditions (0.3). Basically, the proposition 3.4 is equivalent to Lemmata III.4.1, III.9.1, IV.4.1 from [8].

Proposition 3.4. *For given constants $k > 0$ and $\rho > 0$, there exists $\nu_0 = \nu_0(N, g_1, g_0) \in (0, 1)$ such that, if*

$$\text{meas} \{(x, t) \in Q_{k, 2\rho} : u(x, t) < k\} < \nu_0 |Q_{k, 2\rho}|,$$

then

$$\text{ess inf}_{Q_{k, \rho}} u(x, t) \geq \frac{k}{2}.$$

Proof. First, we construct two sequences $\{\rho_n\}_{n=0}^\infty$ and $\{k_n\}_{n=0}^\infty$ such that

$$\rho_n = \rho + \frac{\rho}{2^n} \text{ and } k_n = \frac{k}{2} + \frac{k}{2^{n+1}} \text{ for } n = 0, 1, \dots$$

Because $G(\sigma)$ is an increasing function, a sequence $\{Q_n\}_{n=0}^\infty$ by setting

$$Q_n = K_{\rho_n} \times [-T_{k, \rho_n}, 0]$$

gives nested and shrinking family of cylinders. Let us take a sequence of piecewise linear cutoff functions $\{\zeta_n\}_{n=0}^\infty$ such that

$$\zeta_n = \begin{cases} 1 & \text{inside of } Q_{n+1} \\ 0 & \text{on the parabolic boundary of } Q_n, \end{cases}$$

satisfying

$$|D\zeta_n| \leq \frac{2^{n+1}}{\rho} = \frac{2^{n+1} + 2}{\rho_n},$$

$$0 \leq (\zeta_n)_t \leq \frac{2^{n+g_0+1}}{g_0} k^{-2} G\left(\frac{k}{\rho_n}\right).$$

Particularly, $(\zeta_n)_t$ is driven from the below inequalities;

$$(\zeta_n)_t \leq \left\{ 1 - \left(\frac{\rho_n}{\rho_{n+1}} \right)^{-g_0} \right\}^{-1} \theta k^{-2} G\left(\frac{k}{2\rho_n}\right),$$

$$\frac{\rho_{n+1}}{\rho_n} \geq \frac{\rho/2}{\rho} = \frac{1}{2},$$

$$1 - \left(\frac{\rho_n}{\rho_{n+1}} \right)^{-g_0} = \frac{g_0}{\rho_n^{g_0}} \int_{\rho_{n+1}}^{\rho_n} s^{g_0-1} ds \geq \frac{g_0}{2^{g_0}\rho} (\rho_n - \rho_{n+1}).$$

Note that

$$G(|D\zeta_n|\zeta_n(u - k_n)_-) \leq (2^{n+1} + 2)^{g_1} G\left(\frac{\zeta_n(u - k_n)_-}{\rho_n}\right).$$

Therefore, the local energy estimate (4.1) yields, for some constants γ_0 and γ_1 , that

$$(3.20)$$

$$\begin{aligned} & \sup_t \int_{K_{\rho_n}} G^{r-1} \left(\frac{\zeta_n(u - k_n)_-}{\rho_n} \right) (u - k_n)_-^{s+2} \zeta_n^q dx \\ & + \iint_{Q_n} G(|D(u - k_n)_-|) G^{r-1} \left(\frac{\zeta_n(u - k_n)_-}{\rho_n} \right) (u - k_n)_-^s \zeta_n^q dx dt \\ & \leq \gamma_0 \iint_{Q_n} G^{r-1} \left(\frac{\zeta_n(u - k_n)_-}{\rho_n} \right) (u - k_n)_-^{s+2} \zeta_n^{q-1} (\zeta_n)_t dx dt \\ & + \gamma_1 (2^{n+1} + 2)^{g_1} \iint_{Q_n} G^r \left(\frac{\zeta_n(u - k_n)_-}{\rho_n} \right) (u - k_n)_-^s \zeta_n^{q-1-2g_1} dx dt. \end{aligned}$$

With a property from the level set,

$$(u - k_n)_- = \max\{0, k_n - u\} \leq k_n \leq k,$$

and the setting that $G^{r-1}(\sigma)\sigma^{s+2}, G^r(\sigma)\sigma^s$ are increasing, the right hand side of (3.20) is bounded by

$$(3.21)$$

$$RHS \leq \left\{ \gamma_0 \frac{2^{g_0+n+1}}{g_0\theta} + \gamma_1 (2^{n+1} + 2)^{g_1} \right\} G^r \left(\frac{k}{\rho_n} \right) k^s \iint_{Q_n} \chi_{\{u < k_n\}} dx dt.$$

To find out a lower bound of the left hand side of (3.20), we consider the set $Q_n \cap \{u < k_{n+1}\}$. Indeed in the set $\{u < k_{n+1}\}$, we observe that

$$(u - k_n)_- = \max\{0, k_n - u\} \geq k_n - k_{n+1} = \frac{k}{2^{n+2}},$$

$$(u - k_n)_-^2 G\left(\frac{\zeta_n(u - k_n)_-}{\rho_n}\right)^{-1} \geq 2^{-(n+2)} k^2 G\left(\frac{k}{\rho_n}\right)^{-1} \zeta_n^{-g_1},$$

because σ is an increasing and $\sigma G(\sigma)^{-1}$ is an decreasing functions. Let $u_n := (u - k_n)_-$ for simpler notations. Thus, we obtain that

$$(3.22) \quad \begin{aligned} & 2^{-(n+2)} k^2 G\left(\frac{\zeta_n k}{\rho_n}\right)^{-1} \sup_t \int_{K_{\rho_n}} G^r\left(\frac{\zeta_n u_n}{\rho_n}\right) u_n^s \zeta_n^{q-g_1} dx \\ & + \iint_{Q_n} G(|Du_n|) G^{r-1}\left(\frac{\zeta_n u_n}{\rho_n}\right) u_n^s \zeta_n^q dx dt \\ & \leq \gamma 2^{ng_1} G^r\left(\frac{k}{\rho_n}\right) k^s \iint_{Q_n} \chi_{\{u < k_n\}} dx dt. \end{aligned}$$

To transform the time coordinate, let us say that

$$d_n := k^2 G\left(\frac{k}{\rho_n}\right)^{-1},$$

$$\bar{t} := \frac{t}{d_n}$$

which leads a mapping

$$Q_n \rightarrow \bar{Q}_n = K_{\rho_n} \times [-\theta, 0].$$

Set up

$$(3.23) \quad u(\cdot, t) = \bar{u}(\cdot, t_1 + d_n \bar{t}) \quad \text{and} \quad \zeta_n(\cdot, t) = \bar{\zeta}_n(\cdot, t_1 + d_n \bar{t}).$$

Let $\bar{u}_n := (\bar{u} - k_n)_-$ for simpler notations.

Divide the inequality (3.22) by $2^{-(n+2)} d_n$ and make transformation with respect to the time variable from t to \bar{t} . As a result, we get

$$(3.24) \quad \begin{aligned} & \sup_t \int_{K_{\rho_n} \cap \{\bar{u} < k_{n+1}\}} G^r\left(\frac{\bar{\zeta}_n \bar{u}_n}{\rho_n}\right) \bar{u}_n^s \bar{\zeta}_n^{q-g_1} dx \\ & + \iint_{Q_n \cap \{\bar{u} < k_{n+1}\}} G(|D\bar{u}_n|) G^{r-1}\left(\frac{\bar{\zeta}_n \bar{u}_n}{\rho_n}\right) \bar{u}_n^s \bar{\zeta}_n^q dx d\bar{t} \\ & \leq \gamma 2^{n(g_1+1)} G^r\left(\frac{k}{\rho_n}\right) k^s \iint_{Q_n} \chi_{\{\bar{u} < k_n\}} dx d\bar{t}. \end{aligned}$$

To play with Theorem 4.6, we consider the function

$$v = G^r \left(\frac{\bar{\zeta}_n \bar{u}_n}{2\rho_n} \right) \bar{u}_n^s \bar{\zeta}_n^q.$$

After taking the derivative of v and applying Lemma 1.1, for some constants c_0 and c_1 , we derive

$$|Dv| \leq \frac{c_0}{\rho_n} G(|D\bar{u}_n|) G^{r-1} \left(\frac{\bar{u}_n}{2\rho_n} \right) \bar{u}_n^s + \frac{c_1 2^n}{\rho_n} v.$$

Hence, from the inequality (3.24) and Theorem 4.6, it follows

$$(3.25) \quad \begin{aligned} & \iint_{\bar{Q}_n \cap \{\bar{u} < k_{n+1}\}} G^r \left(\frac{\bar{\zeta}_n \bar{u}_n}{\rho_n} \right) \bar{u}_n^s \bar{\zeta}_n^q dx d\bar{t} \\ & \leq C \rho_n^{-\frac{N}{N+1}} 2^{n(g_1+2)} G^r \left(\frac{k}{\rho_n} \right) k^s \left[\iint_{\bar{Q}_n} \chi_{\{\bar{u} < k_n\}} dx d\bar{t} \right]^{1+\frac{1}{N+1}}. \end{aligned}$$

The left hand side of (3.25) is lower bounded by

$$(3.26) \quad LHS \geq k^s G^{r-1} \left(\frac{k}{\rho_n} \right) G \left(\frac{k_n - k_{n+1}}{\rho_n} \right) \bar{\zeta}_n^{q+rg_0}$$

because $G^{r-1}(\sigma)\sigma^s$ is a nonincreasing function, $G(\sigma)$ is an increasing function, and r is a nonpositive constant. As a result, we now have

$$(3.27) \quad \begin{aligned} & \iint_{\bar{Q}_n} \chi_{\{\bar{u} < k_{n+1}\}} \bar{\zeta}_n^{q+rg_0} dx d\bar{t} \\ & \leq C \rho_n^{-\frac{N}{N+1}} 2^{(n+1)(2g_1+2)} \left[\iint_{\bar{Q}_n} \chi_{\{\bar{u} < k_n\}} dx d\bar{t} \right]^{1+\frac{1}{N+1}}. \end{aligned}$$

Now divide the inequality (3.27) by $|\bar{Q}_n|$ with notice that

$$|\bar{Q}_n| = |K_{\rho_n} \times [-1, 0]| = c\rho_n^N,$$

which is equivalent to

$$\rho_n^{\frac{N}{N+1}} = c|\bar{Q}_n|^{\frac{1}{N+1}}$$

that leads us to the inequality

$$(3.28) \quad \begin{aligned} & \frac{\iint_{\bar{Q}_n} \chi_{\{\bar{u} < k_{n+1}\}} \bar{\zeta}_n^{q+rg_0} dx d\bar{t}}{|\bar{Q}_n|} \\ & \leq C 2^{(n+1)(2g_1+1)} \left[\frac{\iint_{\bar{Q}_n} \chi_{\{\bar{u} < k_n\}} dx d\bar{t}}{|\bar{Q}_n|} \right]^{1+\frac{1}{N+1}}. \end{aligned}$$

We go back to the original time coordinate t from \bar{t} , then we apply below two inequalities,

$$\begin{aligned} \iint_{Q_n} \chi_{\{u < k_{n+1}\}} \zeta_n^{q+rg_0} dx dt &\geq \iint_{Q_{n+1}} \chi_{\{u < k_{n+1}\}} dx dt, \\ |Q_n| &\leq C 2^{N+g_1} |Q_{n+1}|, \end{aligned}$$

to the estimate (3.28).

Eventually, for some constant C , we derive (3.29)

$$\frac{\iint_{Q_{n+1}} \chi_{\{u < k_{n+1}\}} dx dt}{|Q_{n+1}|} \leq C 2^{n(2g_1+1)} \left[\frac{\iint_{Q_n} \chi_{\{u < k_n\}} dx dt}{|Q_n|} \right]^{1+\frac{1}{N+1}}.$$

Applying Lemma 4.7 with

$$\nu_0 := C^{-(N+1)} 2^{-(2g_1+1)(N+1)^2}$$

completes the proof. \square

Proof of main lemma. Now we are ready to prove Lemma 2.1 by applying four propositions in this section.

Proof. We are separating two cases, either $1 < g_0 \leq g_1 \leq 2$ (including singular type equations $1 < g_0 = g_1 < 2$) or $2 \leq g_0 \leq g_1 < \infty$ (including degenerate equations $2 < g_0 = g_1 < \infty$) because of the nature of two cases as appears in Proposition 3.2 and Proposition 3.3 although the way of proofs are very alike each other.

First, we consider the case when $1 < g_0 \leq g_1 \leq 2$. For some constants $M > 0$ and $R > 0$, let us assume that

$$\text{meas} \{(x, t) \in Q_0 : u(x, t) \geq 2M\} \geq \frac{1}{2} |Q_0|$$

where $Q_0 = K_R \times [-T, 0]$ on which T to be determined later. We begin with (2.1), the assumption of Lemma 2.1.

By applying Proposition 3.1 with a fixed constant $\delta_1 = 1/2$, for any constant $\nu_1 \in (0, 1)$, there exist a point $y \in K_R$, a time level $\tau \in [T/16, T]$, and a constant $\eta = \eta(M, \nu_1, \text{data})$ such that $K_{\eta R}^y \subset K_R$ and

$$\text{meas} \{x \in K_{\eta R}^y : u(x, -\tau) < M\} < (1 - \nu_1) |K_{\eta R}^y|.$$

For any $\epsilon \in (0, 1)$, Proposition 3.2 yields that there exists a positive integer $j = j(\nu_1, \epsilon, \text{data})$ such that if

$$\tau \leq (2^{-j} M)^2 G \left(\frac{2^{-j} M}{\eta R} \right)^{-1},$$

then for all $t \in [-\tau, 0]$

$$(3.30) \quad \text{meas} \{x \in K_{\eta R}^y : u(x, t) < 2^{-j}M\} < (1 - (1 - \epsilon)\nu_1) |K_{\eta R}^y|.$$

Therefore, Proposition 3.4 provides that

$$\text{ess} \inf_{Q_1} u(x, t) \geq 2^{-j-1}M$$

where

$$Q_1 = K_{\eta R/2}^y \times [- (2^{-j}M)^2 G \left(\frac{2^{-j}M}{\eta R/2} \right)^{-1}, 0]$$

by fixing $1 - (1 - \epsilon)\nu_1 = \nu_0$ where $\nu_0 = \nu_0(N, g_1, g_0)$ is a fixed constant from Proposition 3.4. For simplicity, let $k = 2^{-j}M$ and $\rho = \eta R/2$. Therefore assumptions on Proposition 3.3 for any $\alpha \in (0, 1)$ hold. For any $\nu \in (0, 1)$, there exists a positive integer $j^* = j^*(N, \alpha, g_1, \eta, \nu)$ such that

$$(3.31) \quad \text{meas} \{(x, t) \in K_R \times [-\tau, 0] : u(x, t) < 2^{-j^*}k\} < \nu |K_R \times [-\tau, 0]|.$$

Now set up $u(x, t) = 2^{-j^*}v(x, t)$, then (3.31) provides

$$\text{meas} \{(x, t) \in K_R \times [-\tau, 0] : v(x, t) < k\} < \nu |K_R \times [-\tau, 0]|$$

that provides a proper assumption to apply Proposition 3.4 for v . Now we claim that Proposition 3.4 holds for v instead of u . We construct a sequences that, for a positive integers $n = 0, 1, 2, \dots$,

$$R_n = R/2 + R/2^{n+1}, \quad k_n = k/2 + k/2^{n+1}$$

and a sequence of cylinders

$$Q_n = K_{R_n} \times [-k_n^2 G \left(\frac{k_n}{R_n} \right)^{-1}, 0].$$

Then let us construct a sequence of a cutoff functions ζ_n such that 1 inside of Q_{n+1} and vanishing out of Q_n satisfying that $|D\zeta_n| \leq 2^{n+2}/R$ and $0 \leq (\zeta_n)_t \leq 2^n k^{-2} G(k/R_n)$. Because Proposition 3.4 relies on the local energy estimate (4.1) for u , we claim that we are able to remove 2^{-j^*} from (4.1) algebraically replacing $(u - k_n)_\pm$ by $2^{-j^*}(v - k_n)_\pm$. Hence Proposition 3.4 holds for v .

Because of the assumption $1 < g_0 \leq g_2 \leq 2$, first integral on the right hand side of (4.1) is handled by following

$$\begin{aligned} & G^{r-1} \left(\frac{2^{-j^*} \zeta_n(v - k_n)_-}{R_n} \right) (2^{-j^*} (v - k_n)_-)^{s+2} \zeta_n^{q-1} (\zeta_n)_t \\ & \leq G^{r-1} \left(\frac{2^{-j^*} k_n}{R_n} \right) (2^{-j^*} k_n)^{s+2} k^{-2} G \left(\frac{k}{R_n} \right) \\ & \leq G^r \left(\frac{2^{-j^*} k_n}{R_n} \right) (2^{-j^*} k_n)^s. \end{aligned}$$

Also a quantity appearing on the left hand of (4.1) is treated as

$$2^{-2j^*} G \left(\frac{k}{R_n} \right) \leq G \left(\frac{2^{-j^*} k}{R_n} \right).$$

Therefore, from Proposition 3.4 gives that

$$\operatorname{ess\,inf}_Q v(x, t) \geq k/2$$

by fixing $\nu = \nu'_0$ where a constant $\nu'_0 = \nu'_0(N, g_1, g_0) \in (0, 1)$ is chosen from Proposition 3.4. Hence we have

$$\operatorname{ess\,inf}_Q u(x, t) \geq 2^{-j^* - j - 1} M$$

where

$$Q = K_{R/2} \times [-\eta^{g_1} k^2 G \left(\frac{k}{R/2} \right)^{-1}, 0].$$

Fianally, we fix

$$T = 16\eta^{g_1} (2^{-j} M)^2 G \left(\frac{2^{-j} M}{R} \right)^{-1}.$$

Second, suppose that $2 \leq g_0 \leq g_1 < \infty$. For some constants $M > 0$ and $R > 0$, let us assume that

$$(3.32) \quad \operatorname{meas} \{(x, t) \in Q_0 : u(x, t) \geq 2M\} \geq \frac{1}{2} |Q_0|$$

where $Q_0 = K_R \times [-T, 0]$ on which T to be determined later which is the assumption of Lemma 2.1.

As the first step, we describe how to obtain strict positiveness of $\operatorname{ess\,inf}_{Q_1} u$ over a certain cylinder Q_1 . Then we extend this result to make our conclusion. By Proposition 3.1 with a fixed constant $\delta_1 = 1/2$, for any constant $\nu_1 \in (0, 1)$, there exist a point $y \in K_R$, a time level $\tau \in [T/16, T]$, and a constant $\eta = \eta(M, \nu_1, \text{data})$ such that $K_{\eta R}^y \subset K_R$ and

$$\operatorname{meas} \{x \in K_{\eta R}^y : u(x, -\tau) < M\} < (1 - \nu_1) |K_{\eta R}^y|.$$

Then for any $\epsilon \in (0, 1)$, Proposition 3.3 yields that there exists a positive integer $j = j(\nu_1, \epsilon, \text{data})$ such that if

$$\tau \leq M^2 G \left(\frac{M}{\eta R} \right)^{-1},$$

then for all $t \in [-\tau, 0]$

$$(3.33) \quad \text{meas} \{x \in K_{\eta R}^y : u(x, t) < 2^{-j} M\} < (1 - (1 - \epsilon)\nu_1) |K_{\eta R}^y|.$$

Now we take subdivision of the cube $K_{\eta R}^y$ into 2^{lN} congruent subcylinders, then for any positive integer l , there exist some subcylinders such that

$$\text{meas} \left\{ x \in K_{2^{-l}\eta R}^{y'} : u(x, t) < 2^{-j} M \right\} < (1 - (1 - \epsilon)\nu_1) |K_{2^{-l}\eta R}^{y'}|.$$

Otherwise (3.33) fails. In particular, we choose l to be the least integer such that

$$l \geq \frac{(g_1 - 2)j}{g_0}$$

for satisfying

$$(2^{-j} M)^2 G \left(\frac{2^{-j} M}{2^{-l}\eta R} \right)^{-1} \leq M^2 G \left(\frac{M}{\eta R} \right)^{-1}.$$

Therefore, Proposition 3.4 (by choosing $(1 - \epsilon)\nu_1 = 1 - \nu_0$ where ν_0 is given in Proposition 3.4) implies that

$$(3.34) \quad \text{ess inf}_{Q_1} u(x, t) \geq 2^{-j-1} M$$

where

$$Q_1 = K_{2^{-l-1}\eta R}^{y'} \times \left[- (2^{-j} M)^2 G \left(\frac{2^{-j} M}{2^{-l-1}\eta R} \right)^{-1}, 0 \right].$$

Then we note that the constant M in the assumption (3.32) can be replaced by σM for any $\sigma \in (0, 1)$. Therefore the first step for obtaining Q_1 provides that

$$(3.35) \quad \text{ess inf}_{Q_2} u(x, t) \geq 2^{-j-1} \sigma M$$

where

$$Q_2 = K_{2^{-l-1}\eta R}^{y'} \times \left[- (2^{-j} \sigma M)^2 G \left(\frac{2^{-j} \sigma M}{2^{-l-1}\eta R} \right)^{-1}, 0 \right].$$

For simplicity, denote that $\rho = 2^{-l-1}\eta R$ and $k = 2^{-j} M$. Moreover we let $u(x, t) = \sigma v(x, t)$ and

$$t^* = (2^{-l}\eta)^{g_1} (\sigma k)^2 G \left(\frac{\sigma k}{\rho} \right)^{-1} \leq (\sigma k)^2 G \left(\frac{\sigma k}{\rho} \right)^{-1}.$$

Because l and η are fixed constants, we are applying Proposition 3.3 to $(u - k_i)_\pm = \sigma(v - k_i)_\pm$ where $k_i = 2^{-i}k$ for $i = 0, 1, \dots$. Then σ would disappear from the energy estimates. We set ζ to be a cutoff function vanishing outside of $Q_3 = K_R \times [-t^*, 0]$ and $\zeta = 1$ in the cylinder $Q_4 = K_{R/2} \times [-t^*/2, 0]$ satisfying $|D\zeta| \leq 2/R$ and $0 \leq \zeta_t \leq 2/t^*$. Then (4.1) implies

$$\begin{aligned} & \iint_{Q_3} G(\sigma|D(v - k_i)_\pm|) G^{r-1} \left(\frac{\zeta\sigma(v - k_i)_\pm}{R} \right) \sigma^s(v - k_i)_\pm \zeta^q dx dt \\ & \leq c_1 \iint_{Q_3} G^{r-1} \left(\frac{\zeta\sigma(v - k_i)_\pm}{R} \right) \sigma^{s+2}(v - k_i)_\pm \zeta^{q-1} \zeta_t dx dt \\ & \quad + c_2 \iint_{Q_3} G(|D\zeta|\zeta\sigma(v - k_i)_\pm) G^{r-1} \left(\frac{\zeta\sigma(v - k_i)_\pm}{R} \right) \sigma^s(v - k_i)_\pm \zeta^{q-1-2g_1} dx dt. \end{aligned}$$

Since $(v - k_i)_- \leq k_i$ and two increasing mappings $w \rightarrow G^r(w)w^s$ and $w \rightarrow G^{r-1}(w)w^{s+2}$ for the right hand side and a decreasing mapping $w \rightarrow G^{r-1}(w)w^s$ to handle the left hand side, the previous inequality is reduced to

$$\iint_{Q_3} G(\sigma|D(v - k_i)_-|) \chi_{\{k_{i+1} \leq v \leq k_i\}} \zeta^q dx dt \leq CG \left(\frac{\sigma k_i}{R} \right) |K_R \times [-t^*, 0]|$$

which appears in the proof of Proposition 3.3, (3.12). Indeed, we can apply Proposition 3.3 to v . Therefore, for any $\nu \in (0, 1)$, there exists a positive integer $j^* = j^*(N, \eta, \nu, \text{data})$ such that

$$\text{meas} \{ (x, t) \in Q_4 : v(x, t) < 2^{-j^*}k \} < \nu|Q_4|.$$

Recall that

$$Q_4 = K_{R/2} \times \left[-\frac{1}{2} (2^{-l}\eta)^{g_1} (\sigma k)^2 G \left(\frac{\sigma k}{R} \right)^{-1}, 0 \right].$$

We fix $\nu = \nu'_0$ where ν'_0 is given in Proposition 3.4 and also let $\sigma = 2^{-j^*}$. Therefore, we are able to apply Proposition 3.4 (we use $\sigma(v - k_n)_\pm$ in the place of $(u - k_n)_\pm$ from the energy estimate (4.1), then the time setting $(\sigma k_n)G(\sigma k_n/R_n)^{-1}$ provides the proper time length) to derive conclusion that

$$\text{ess} \inf_Q v(x, t) \geq 2^{-j^*-1}k$$

which means

$$\text{ess} \inf_Q u(x, t) \geq 2^{-2j^*-j-1}M$$

where

$$Q = K_{R/4} \times \left[- (2^{-l}\eta)^{g_1} (2^{-j^*-j}M)^2 G \left(\frac{2^{-j^*-j}M}{R/2} \right)^{-1}, 0 \right].$$

We fix

$$T = 16\eta^{g_1} (2^{-j^*-j}M)^2 G \left(\frac{2^{-j^*-j}M}{R} \right)^{-1}.$$

□

4. PROOF FOR AUXILIARY THEOREMS

4.1. The local energy estimate. The local energy estimate is one of fundamental inequality playing important roles, especially Proposition 3.1, Proposition 3.2, and Proposition 3.3. The inequality is equivalent to one appearing on Section II.3-(i) from [8] and same as Proposition 2.4 from [22] if $g_0 = g_1 = p$. Some techniques come from Section 3 in [18].

Proposition 4.1. *Let u be a locally bounded weak solution of (0.1) with structure conditions (0.3) in a cylinder $Q_\rho := K_\rho \times [t_0, t_1]$. Then there exist constants c_0 , c_1 , and c_2 depending on data such that*

$$\begin{aligned} (4.1) \quad & \sup_t \int_{K_\rho} G^{r-1} \left(\frac{\zeta(u-k)_\pm}{\rho} \right) (u-k)_\pm^{s+2} \zeta^q dx \\ & + c_0 \iint_{Q_\rho} G(|D(u-k)_\pm|) G^{r-1} \left(\frac{\zeta(u-k)_\pm}{\rho} \right) (u-k)_\pm^s \zeta^q dx dt \\ & \leq c_1 \iint_{Q_\rho} G^{r-1} \left(\frac{\zeta(u-k)_\pm}{\rho} \right) (u-k)_\pm^{s+2} \zeta^{q-1} \zeta_t dx dt \\ & + c_2 \iint_{Q_\rho} G(|D\zeta| \zeta(u-k)_\pm) G^{r-1} \left(\frac{\zeta(u-k)_\pm}{\rho} \right) (u-k)_\pm^s \zeta^{q-1-2g_1} dx dt \end{aligned}$$

with a cutoff function $0 \leq \zeta \leq 1$ on the cylinder Q_ρ where

$$(4.2) \quad r = \frac{1-g_1}{g_0}, \quad s = g_1 + \frac{g_1}{g_0}(g_1-1), \quad \text{and} \quad q = 2+2g_1 + \frac{g_1}{g_0}(g_1-1).$$

Proof. We assume that u is differentiable in terms of the time variable. Such an assumption is removed by applying Steklov average.

The choices (4.2) is made to satisfy that

$$\begin{aligned}
(r-1)g_0 + (s+1) &> 0, \\
rg_0 + s &> 0, \\
(r-1)g_1 + (s+1) &> 0, \\
(r-1)g_1 + q &> 0, \\
q - 1 - 2g_1 + rg_1 &\geq 0, \\
(r-1)g_1 + q &> (r-1)g_0 + (s+1), \\
(r-1)g_1 + s &\leq 0.
\end{aligned}$$

The first six inequalities are generated to derive the local energy estimate properly. The first three inequalities imply that $G^{r-1}(\sigma)\sigma^{s+2}$ and $G^r(\sigma)\sigma^s$ are increasing functions. The last inequality is driven so that $G^{r-1}(\sigma)\sigma^s$ is nonincreasing that supports simpler calculations.

For a bounded weak supersolution u , let the test function to be

$$\varphi(x, t) = G^{r-1} \left(\frac{\zeta(u-k)_-}{\rho} \right) (u-k)_-^{s+1} \zeta^q,$$

with a piecewise linear cutoff function ζ vanishing on the parabolic boundary of Q_ρ . For simpler notation, let $\bar{u} := (u-k)_-$. Then we have

$$\begin{aligned}
D\varphi &= \left\{ (r-1) \frac{\zeta \bar{u}}{\rho} g \left(\frac{\zeta \bar{u}}{\rho} \right) + (s+1) G \left(\frac{\zeta \bar{u}}{\rho} \right) \right\} G^{r-2} \left(\frac{\zeta \bar{u}}{\rho} \right) \bar{u}^s \zeta^q D\bar{u} \\
&\quad + \left\{ (r-1) \frac{\zeta \bar{u}}{\rho} g \left(\frac{\zeta \bar{u}}{\rho} \right) + q G \left(\frac{\zeta \bar{u}}{\rho} \right) \right\} G^{r-2} \left(\frac{\zeta \bar{u}}{\rho} \right) \bar{u}^{s+1} \zeta^{q-1} D\zeta.
\end{aligned}$$

It follows that

$$\begin{aligned}
(4.3) \quad &\iint_{Q_\rho} \mathbf{A}(x, t, u, Du) \cdot D\varphi \, dx \, dt \\
&\geq \{(r-1)g_0 + (s+1)\} C_0 \iint_{Q_\rho} G(|Du|) G^{r-1} \left(\frac{\zeta \bar{u}}{\rho} \right) \bar{u}^s \zeta^q \, dx \, dt \\
&\quad - \{(r-1)g_1 + q\} C_1 \iint_{Q_\rho} g(|Du|) |D\zeta| G^{r-1} \left(\frac{\zeta \bar{u}}{\rho} \right) \bar{u}^{s+1} \zeta^{q-1} \, dx \, dt.
\end{aligned}$$

Owing to Lemma 1.1, set $\sigma_1 = |D\zeta|\bar{u}/\zeta$ and $\sigma_2 = |Du|$. Then we obtain, for any $\epsilon_1 > 0$, that

$$\begin{aligned}
 & G^{r-1} \left(\frac{\zeta \bar{u}}{\rho} \right) \bar{u}^s \zeta^q g(|Du|) \frac{|D\zeta \bar{u}|}{\zeta} \\
 (4.4) \quad & \leq \epsilon_1 g_1 G(|Du|) G^{r-1} \left(\frac{\zeta \bar{u}}{\rho} \right) \bar{u}^s \zeta^q \\
 & \quad + \epsilon_1^{1-g_1} g_1 G \left(\frac{|D\zeta| \bar{u}}{\zeta} \right) G^{r-1} \left(\frac{\zeta \bar{u}}{\rho} \right) \bar{u}^s \zeta^{q-2g_1}.
 \end{aligned}$$

Now, by setting

$$F(\bar{u}) = \int_0^{\bar{u}} G^{r-1} \left(\frac{\zeta \alpha}{\rho} \right) \alpha^{s+1} d\alpha,$$

it yields

$$\begin{aligned}
 & \iint_{Q_\rho} u_t \varphi(x, t) dx dt \\
 (4.5) \quad & = \int_{K_\rho} F(\bar{u}) \zeta^q dx \Big|_{t_0}^{t_1} - q \iint_{Q_\rho} F(\bar{u}) \zeta^{q-1} \zeta_t dx dt.
 \end{aligned}$$

To estimate bounds for the function $F(\bar{u})$, first we derive

$$(4.6) \quad F''(\bar{u}) \bar{u} \leq \{(r-1)g_1 + (s+1)\} F'(\bar{u}),$$

$$(4.7) \quad F''(\bar{u}) \bar{u} \geq \{(r-1)g_0 + (s+1)\} F'(\bar{u}).$$

From the integration by parts, we get

$$(4.8) \quad F(\bar{u}) = \int_0^{\bar{u}} F'(\sigma) d\sigma = \bar{u} F'(\bar{u}) - \int_0^{\bar{u}} \sigma F''(\sigma) d\sigma.$$

The equation (4.8) applying (4.6) or (4.7) generates

$$\begin{aligned}
 F(\bar{u}) & \geq \frac{1}{(r-1)g_1 + (s+2)} \bar{u} F'(\bar{u}), \\
 F(\bar{u}) & \leq \frac{1}{(r-1)g_0 + (s+2)} \bar{u} F'(\bar{u}),
 \end{aligned}$$

where

$$F'(\bar{u}) = G^{r-1} \left(\frac{\zeta \bar{u}}{\rho} \right) \bar{u}^{s+1}.$$

First, we apply (4.4) with ϵ_1 such that

$$\epsilon_1 < \frac{C_0 \{(r-1)g_1 + (s+1)\}}{g_1 \{(r-1)g_0 + q\}}$$

to the inequality (4.3), then combine with the inequality (4.5).

For a bounded weak subsolution u , similar story works with the test function

$$\varphi(x, t) = G^{r-1} \left(\frac{\zeta(u-k)_+}{\rho} \right) (u-k)_+^{s+1} \zeta^q,$$

□

4.2. The logarithmic energy estimate. The logarithmic energy estimate (4.9) which is used to prove Proposition 3.2. The estimate is modified from one in Section II.3-(ii) in [8] and similar to the logarithmic estimate in Section 3.3 from [22]. The functions h and H are defined on Lemma 1.2.

Proposition 4.2. *Let u be a local bounded weak solution of (0.1) under 0.3 in a cylinder $K_R \times [t_0, t_1]$ and $k \in \mathbb{R}$. For $q \geq g_1$, then we have*

$$(4.9) \quad \int_{K_R \times \{t_1\}} H(\Psi^2) \zeta^q dx \leq \int_{K_R \times \{t_0\}} H(\Psi^2) \zeta^q dx + C_1 C_0^{1-g_1} (2qg_1)^{g_1} \int_{t_0}^{t_1} \int_{K_R} h(\Psi^2) |\Psi| |\Psi'|^2 G \left(\frac{|D\zeta|}{|\Psi'|} \right) \zeta^{q-g_1} dx dt$$

where

$$\Psi(u) = \ln^+ \left[\frac{k}{(1+\delta)k \pm (u-k)_\pm} \right]$$

with for some constant k and $\delta \in (0, 1)$.

Proof. We assume that u is differentiable in terms of the time variable. Such an assumption is removed by applying Steklov average.

Suppose that u is a bounded weak supersolution. For some $k > 0$, define the test function

$$\varphi = 2h(\Psi^2) \Psi \Psi' \zeta^q$$

where

$$\Psi(u) = \ln^+ \left[\frac{k}{(1+\delta)k \pm (u-k)_\pm} \right]$$

with a constant $\delta \in (0, 1)$ and $q > g_1$ and a cut-off function ζ independent of the time variable ($\zeta_t = 0$). Then

$$\Psi'(u) = \frac{-1}{(1+\delta)k \pm (u-k)_\pm}, \quad \Psi''(u) = \frac{1}{[(1+\delta)k \pm (u-k)_\pm]^2} = (\Psi')^2.$$

For a nonnegative solution u , we have $0 \leq (u-k)_- \leq k$ that implies

$$\frac{1}{(1+\delta)k} \leq \Psi' \leq \frac{1}{\delta k}, \quad 0 \leq \Psi \leq \ln^+ \frac{1}{\delta}.$$

Then $2h(\Psi^2)\Psi\Psi' \in L^\infty$ and

$$[2h(\Psi^2)\Psi\Psi']' \leq [4(g_1 - 1) + 2]h(\Psi^2)(\Psi')^2 + 2h(\Psi^2)\Psi(\Psi')^2 \in L^\infty.$$

Therefore, $\varphi(u)$ is an admissible test function.

First, the time derivative part generates

$$\begin{aligned} & \int_{t_0}^{t_1} \int_{K_R} u_t 2h(\Psi^2) \Psi \Psi' \zeta^q dx dt \\ &= \int_{t_0}^{t_1} \int_{K_R} \left[\frac{d}{dt} H(\Psi^2) \right] \zeta^q dx dt \\ &= \int_{K_R \times \{t_1\}} H(\Psi^2) \zeta^q dx - \int_{K_R \times \{t_0\}} H(\Psi^2) \zeta^q dx. \end{aligned}$$

Second, we study the derivative of the test function that

$$\begin{aligned} D\varphi &= 4h'(\Psi^2)(\Psi\Psi')^2\zeta^q Du + 2h(\Psi^2)(\Psi')^2\zeta^q Du \\ &\quad + 2h(\Psi^2)\Psi\Psi''\zeta^q Du + 2qh(\Psi^2)\Psi\Psi'\zeta^{q-1}D\zeta. \end{aligned}$$

Owing to (c) of Lemma 1.2, we estimate

$$D\varphi \geq [\{4(g_0 - 1) + 2\} + 2\Psi] h(\Psi^2)(\Psi')^2\zeta^q Du + 2qh(\Psi^2)\Psi\Psi'\zeta^{q-1}D\zeta.$$

It follows that

$$\begin{aligned} & \int_{t_0}^{t_1} \int_{K_R} \mathbf{A}(x, t, u, Du) \cdot D\varphi dx dt \\ (4.10) \quad & \geq C_0 \int_{t_0}^{t_1} \int_{K_R} G(|Du|)h(\Psi^2) [4(g_0 - 1) + 2 + \Psi] (\Psi')^2 \zeta^q dx dt \\ & \quad - 2qC_1 \int_{t_0}^{t_1} \int_{K_R} g(|Du|)h(\Psi^2)\Psi\Psi'\zeta^{q-1}|D\zeta| dx dt. \end{aligned}$$

Young's inequality, Lemma 1.1, tells us that

$$\begin{aligned} & g(|Du|)h(\Psi^2)\Psi\Psi'\zeta^{q-1}|D\zeta| \\ &= h(\Psi^2)\Psi(\Psi')^2\zeta^q g(|Du|)(\Psi')^{-1}\zeta^{-1}|D\zeta| \\ &\leq h(\Psi^2)\Psi(\Psi')^2\zeta^q \left\{ \epsilon_2 g_1 G(|Du|) + \epsilon_2 g \left(\frac{|D\zeta|}{\epsilon_2 \Psi' \zeta} \right) \frac{|D\zeta|}{\epsilon_2 \Psi' \zeta} \right\} \end{aligned}$$

for any $\epsilon_2 > 0$. Particularly choosing ϵ_2 to satisfy $C_0 = 2qg_1\epsilon_2$ leads to (4.9). \square

4.3. Collecting positivity.

Lemma 4.3. *Let $u(\cdot, \tau) \in W^{1,1}(K_\rho)$ for all τ and satisfy*

$$\int_{K_\rho \times \{\tau\}} |Du| dx \leq \gamma \rho^{N-1} \text{ and } \text{meas} \{x \in K_\rho : u(x, \tau) > 1\} \geq \alpha |K_\rho|$$

for some $\gamma > 0$ and $\alpha \in (0, 1)$. Then for every $\delta \in (0, 1)$ and $0 < \lambda < 1$, there exist $x_0 \in K_\rho$ and $\eta = \eta(\alpha, \delta, \gamma, \lambda, N) \in (0, 1)$ such that

$$\text{meas} \{x \in K_{\eta\rho}^{x_0} : u(x, \tau) > \lambda\} > (1 - \delta) |K_{\eta\rho}^{x_0}|.$$

This lemma is from [9].

4.4. Poincare type inequality.

Theorem 4.4. *Suppose that $u \in W^{1,1}(K_\rho)$ with $u(x) = 0$ on some set Σ_0 of positive measure. Then for any measurable set Σ from K_ρ , the inequality holds*

$$\int_{\Sigma} u(x) \varphi(x) dx \leq \beta \frac{\rho^N}{|\Sigma_0|} |\Sigma|^{\frac{1}{N}} \int_{K_\rho} |Du(x)| \varphi(x) dx.$$

This theorem is appearing on Section 2.5 from [16].

Corollary 4.5. *Let $v \in W^{1,1}(K_\rho^{x_0}) \cap C(K_\rho^{x_0})$ for some $\rho > 0$ and some $x_0 \in \mathbb{R}^N$ and let k and l be any pair of real numbers such that $k < l$. Then there exists a constant γ depending only upon N, p and independent of k, l, v, x_0, ρ , such that*

$$\begin{aligned} & (l - k) \text{meas} \{x \in K_\rho^0 : v(x) < l\} \\ & \leq \gamma \frac{\rho^{N+1}}{\text{meas} \{x \in K_\rho^0 : v(x) > k\}} \int_{\{x \in K_\rho^0 : k < v(x) < l\}} |Dv| dx. \end{aligned}$$

This lemma is appearing on page 5 from [8].

4.5. Embedding theorem.

Theorem 4.6. *For a nonnegative function $v \in W_0^{1,1}(Q)$ where $Q = K \times [t_0, t_1]$, $K \subset \mathbb{R}^N$, we have*

$$\begin{aligned} (4.11) \quad \iint_Q v dx dt & \leq C(N) \left[\iint_Q \chi_{\{v>0\}} dx dt \right]^{\frac{1}{N+1}} \times \\ & \left[\text{ess sup}_{t_0 \leq t \leq t_1} \int_K v dx \right]^{\frac{1}{N+1}} \cdot \left[\iint_Q |Dv| dx dt \right]^{\frac{N}{N+1}}. \end{aligned}$$

Proof. First, by the Hölder inequality, we obtain

$$(4.12) \quad \iint_Q v \, dx \, dt \leq \left[\iint_Q \chi_{\{v>0\}} \, dx \, dt \right]^{\frac{1}{N+1}} \cdot \left[\iint_Q v^{\frac{N+1}{N}} \, dx \, dt \right]^{\frac{N}{N+1}}.$$

Second, by Hölder inequality and Sobolev inequality for $p = 1$, we have

$$(4.13) \quad \begin{aligned} \int_K v^{\frac{N+1}{N}} \, dx &\leq \left[\int_K v^{\frac{N}{N-1}} \, dx \right]^{\frac{N-1}{N}} \cdot \left[\int_K v \, dx \right]^{\frac{1}{N}} \\ &\leq C(N) \int_K |Dv| \, dx \cdot \left[\int_K v \, dx \right]^{\frac{1}{N}}. \end{aligned}$$

Combining two inequalities (4.12) and (4.13) produces the inequality (4.11). \square

4.6. Iteration.

Lemma 4.7. *Let $\{Y_n\}$, $n = 0, 1, 2, \dots$, be a sequence of positive numbers, satisfying the recursive inequalities*

$$Y_{n+1} \leq Cb^n Y_n^{1+\alpha}$$

where $C, b > 1$ and $\alpha > 0$ are given numbers. If

$$Y_0 \leq C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}},$$

then $\{Y_n\}$ converges to zero as $n \rightarrow \infty$.

This lemma is on Section I.4 from [8].

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